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ABSTRACT

In this paper, the time-fractional nonlinear dispersive (TFND) partial differential equations (PDEs) in the sense of conformable fractional derivative (CFD) are proposed and analyzed. Three types of TFND partial differential equations are considered in the sense of CFD, which are the TFND Boussinesq, TFND Klein-Gordon, and TFND B(2, 1, 1) PDEs. Solitary pattern solutions for this class of TFND partial differential equations based on the residual fractional power series method is constructed and discussed. Numerical and graphical results are also provided and conferred quantitatively to clarify the required solutions. The results suggest that the algorithm presented here offers solutions to problems in a rapidly convergent series leading to ideal solutions. Furthermore, the results obtained are like those in previous studies that used other types of fractional derivatives. In addition, the calculations used were much easier and shorter compared with other types of fractional derivatives.

The current article presents solitary solutions for the time-fractional nonlinear dispersive (TFND) partial differential equations (PDEs) in the sense of conformable fractional derivative (CFD). The residual power series (RPS) technique is used to determine the coefficients values of the fractional series solution.

Three types of TFND partial differential equations are considered in the sense of CFD, which are the TFND Boussinesq, TFND Klein-Gordon, and TFND B(2, 1, 1) PDEs to test and verify the effectiveness of the used method.

The contributions of the utilized analysis can be summarized in the form of the following salient features:

- In this analysis, we introduce a new form of fractional power series (FPS) and a new form of fractional Taylor’s series in the sense of CFD.
- We use the proposed new form of fractional power series to construct solitary solutions for the TFND partial differential equations in the sense of CFD.
- The RPS method is developed in order to find solitary solutions for this type of PDEs.
- Three types of TFND partial differential equations are considered in the sense of CFD to verify the effectiveness of presented method.
- Numerical and graphical results are also provided and conferred quantitatively to clarify the required solutions.
- The results obtained are like those in previous studies that used other types of fractional derivatives. Furthermore, the calculations used were much easier and shorter compared with other types of fractional derivatives.
• Since there are many advantages to the CFD definition, the use of CFD in PDE modeling can be an appropriate substitute for other types of fractional derivatives.
• We observed that the RPS method is very suitable, easy, and effective to solve the PDEs and can be used to solve other types of differential equations.

I. INTRODUCTION

Many real-life phenomena are converted into mathematical models through partial differential equations (PDEs) of fractional order. It is generally advised to utilize this formulation in order to conserve the original systems behavior. Certainly, this will aid to enhance understanding of the real-life phenomena and decrease computational difficulty without losing any of the original behaviors. Recently, fractional DEs have been extensively employed to illustrate numerous phenomena in many applications such as in the control theory, fluid flow, diffusion problem for dispersive transport in amorphous semiconductors, liquid crystals, biosystems, electrodynamics, electrostatics and elasticity, systems identification, signal processing, biological processes, and finance. The nonlocal property of the fractional DEs is one of the most significant advantages for using it in these applications. It is known that the differential and integral operator of integer order are local; nevertheless, the differential and integral operators of fractional order are non-local, implying that a system next state depends on its current and historical states. Moreover, the excellent explanation of several models in engineering, physics, and mathematics using fractional differential operators is related to this reason.

Solitary approximation solutions of the time-fractional nonlinear dispersive (TFND) PDEs (TFND-PDEs) have been investigated broadly in many papers due to them being significant and substantial in understanding the physical phenomena. However, various techniques have been suggested to find the solution of the TFND-PDEs, such as the invariant subspace method, variational iteration method, and homotopy perturbation method. However, the solution of the TFND-PDEs of third order is proposed using the nonlinear capacity and eigenfunction technique, the fractional variation iteration method, fractional differential transform method, and other methods. Moreover, the fractal dimension of solutions of dispersive PDE and various features of this problem was studied by many researchers where their techniques were applied to Boussinesq, Schrodinger, the fractional Schrodinger, airy, and the gravity capillary water wave equations.

The residual fractional power series (RPS) method is an efficient technique to obtain the coefficients of the power series expansion for nonlinear and linear categories of DEs without perturbation or discretization and was originally suggested by Abu Arqub in Ref. 35 to solve the ordinary differential equations (ODEs) of ordinary derivative; then, it was developed by El-Ajou in collaboration with Arqub, to solve ODEs and PDEs of fractional order (this is for scientific secretariat). The RPS method is adopted in this work in order to obtain the analytic solitary solution for TFND-PDEs. This solution is gained using truncated series solution and also his solution and its derivatives of fractional order are valid for each random point. On the other hand, to switch from lower to higher order and from linear to nonlinear, the RFPS does not need any converting; consequently, the RFPS can be employed to the problem directly by selecting a suitable approximation of the initial guess and for more details, one can refer to Refs. 35–41.

In the past few decades, the scientists have depended only on analytical methods to find the solutions of such problems in science fields. Nevertheless, the analytical solutions are limited in practice to linear and low dimensional models and therefore cannot be apply often to solve nonlinear problem. Nowadays, numerical techniques and computers offer easy way to solve complicated problems. Consequently, the focus on formulating and interpreting the problem has become more important than solving it. Generally, it is not straightforward to obtain the exact solitary solutions of TFND-PDEs, although such solutions are constantly required due to their benefits in physics. However, several physical phenomena cannot be solved analytically. So, researchers have given more attention to obtain the exact solutions numerically to the TFND-PDEs using numerical methods. Yet, some of these methods and their applications can be found in the literature.

Our purpose of the present paper is to find exact (numerical) solitary pattern solutions using the RPS analytical method for the TFND-PDEs in the sense of the conformable fractional derivative (CFD), which are given in the form of the following models:

- The TFND Boussinesq equation:
  \[ T^\alpha_t \psi(t, r) - (\psi^2(r, t))_{rr} - \psi_r(r, t) + (\psi(r, t)u_t(r, t))_r = 0, \]
  \[ 0 < \alpha \leq 1, \quad r \in \Omega, \quad t > s \geq 0. \]  

- The TFND Klein-Gordon equation:
  \[ T^\alpha_t \psi(t, r) - \lambda(\psi^2(r, t))_{rr} + \kappa(\psi^2(r, t))_{rrr} = 0, \]
  \[ 0 < \alpha \leq 1, \quad r \in \Omega, \quad t > s \geq 0. \]  

- The TFND \( B(2, 1, 1) \) equation:
  \[ T^\alpha_t \psi(t, r) + \lambda(\psi^2(r, t))_{rr} - \kappa(\psi(r, t)\psi_r(r, t))_r = 0, \]
  \[ 0 < \alpha \leq 1, \quad r \in \Omega, \quad t > s \geq 0, \]  

subject to
  \[ \psi(r, s) = f(r), \quad r \in \Omega, \]
  \[ T^\alpha_s \psi(r, s) = g(r), \quad 0 < \alpha \leq 1, \quad r \in \Omega, \]  

where \( \lambda, \kappa > 0 \) are constants, \( T^\alpha_s \) is the conformable fractional derivative of order \( \alpha > 0 \) in which \( T^\alpha_1 = T^\alpha \), the sequential derivative, \( \psi(r, t) \) is a multivariable function that can be expanded in fractional power series (FPS) form, and \( f(r), g(r) \) are functions of \( r \in \Omega \). For \( \alpha = 1 \), then the TFND-PDEs that are mentioned in Eqs. (1.1)–(1.3) reduce to the usual Boussinesq, Klein-Gordon, and \( B(2, 1, 1) \) equations, respectively. These equations are playing very important roles in physics applications and many real-life problems. For instance, the Klein-Gordon equations are applied in absorbing media, dislocations propagation, and the motion of suspended cables and inca rope suspension bridges. The \( B(2, 1, 1) \) equations are implemented to study shallow-water waves, optical solitons, and density waves in traffic flow. The Boussinesq equations can be employed to explain nonlinear beams small oscillations, shallow-water waves, long waves, and nonlinear atomic chains.
This paper is arranged as follows: Definitions and some essential results for the conformable fractional calculus are given in the next section (Sec. II). In Sec. III, the RFPS method is applied to create a pattern of solutions for TFNDE-PDEs where this process is presented in algorithm. In Sec. IV, the simplicity and capability of the proposed method is demonstrated via numerical simulations of some applications. Finally, the conclusion is given in Sec. V.

II. OVERVIEW OF CONFORMABLE FRACTIONAL CALCULUS

There are several definitions of fractional differentiation and integration, such as Caputo’s, Riemann-Liouville’s, and Grünwald-Letnikov’s definitions. A few years ago, there was a new definition of fractional derivatives that was based on the concept of limit and called the conformable fractional derivative (CFD). Several fractional differential equations have been reworked using the CFD concept, for example, heat differential equation, Bernoulli and Riccati differential equations, Hamiltonian systems, Calogero-Bogoyavlenskii-Schiff equation in (2+1) dimensions, and KdV-Burgers equation and have been solved by analytical and numerical methods. For this research purpose, the definition of CFD is used to allow us to construct the solution of TFNDE-PDEs and the initial conditions take the classical form that is used in the cases of ordinary derivatives.

For easy reading and follow-up, various preliminary results and definitions of CFDs and integrals are given below. For more details, reader can refer to Refs. 42 and 43.

**Definition 2.1:** For \( n = [\alpha] \) (the ceiling of \( \alpha \)), the time-CFD of order \( \alpha \) of \( \psi(t) \) is defined as

\[
\mathcal{T}_t^\alpha \psi(t) = \psi_t^{(\alpha)}(t) = \frac{\partial^\alpha \psi (t)}{\partial t^\alpha} = \lim_{\epsilon \to 0} \frac{\psi_{r,t}^{(n-1)} (t, r \in \Omega, t > s, (2.1)

and \( \psi_t^{\alpha} (r,s) = \lim_{t\to s} \psi_t^{(\alpha)}(r,t), r \in \Omega \) provided the limit exists. If the time-CFD of order \( \alpha \) of \( \psi \) exists, then we simply say that \( \psi \) is \( \alpha \)-differentiable with respect to \( t \).

**Theorem 2.1:** Let \( \alpha \in (0, 1] \) and \( \psi(t), \phi(t), r \in \Omega, t > s \) be \( \alpha \)-differentiable with respect to \( t \) at a point \( t > s \). Then,

1. \( \frac{\partial^{\alpha} (a \psi + b \phi)}{\partial t^{\alpha}} = a \psi_t^{(\alpha)} + b \phi_t^{(\alpha)} \), \( a, b \) are constants.
2. \( \frac{\partial^\alpha (t-s)^p}{\partial t^\alpha} = p(t-s)^{p-\alpha} \) for all \( p \in \mathbb{R} \),
3. \( \frac{\partial^{\alpha} (\psi \phi)}{\partial t^{\alpha}} = \psi \phi_t^{(\alpha)} + \phi \psi_t^{(\alpha)} \).
4. \( \frac{\partial^{\alpha} (\psi)}{\partial t^{\alpha}} = \frac{\partial^{\alpha} \psi}{\partial t^{\alpha}} - \phi \psi_t^{(\alpha)} \)
5. \( \frac{\partial^{\alpha} \psi (t)}{\partial t^{\alpha}} = (t-s)^{1-\alpha} \frac{\partial^{\alpha} \psi (t)}{\partial s^{\alpha}} \)

**Remark 2.1:** (i) It should be noted here that when \( \alpha \in (n-1, n] \), it is easy to prove that

(1) \( \frac{\partial^\nu \psi (t)}{\partial t^\nu} (t-s)^{\alpha-n} \frac{\partial^n \psi (t)}{\partial t^n}, t > 0 \),
(2) \( \frac{\partial^\alpha \psi (t-s)^{\nu}}{\partial t^{\nu}} (t-s)^{\alpha-n} \), \( \alpha \in (n-1, n] \), \( n \in \mathbb{N} \), \( \nu \in \mathbb{N} \), \( \psi(t) \) is \( \nu \)-differentiable with respect to \( t \).
(3) \( \frac{\partial^\alpha \psi (t-s)^{\nu}}{\partial t^{\nu}} (t-s)^{\alpha-n}, p \neq n-1, n-2, n-3, \ldots \)

(ii) If \( \psi (t) \) is \( n \)-times differentiable with respect to \( t \) for \( t > s \), then it is \( \alpha \)-differentiable with respect to \( t \) for \( t > s \).

**Definition 2.2:** The \( n \)-times conformable fractional integral (CFI) of order \( \alpha \in (n-1, n] \) of \( \psi (t) \) is defined as

\[
I_t^\alpha \psi (t) = \frac{1}{(n-1)!} \int_s^t (t-x)^{n-1}\psi (x) (t-x)^{-\alpha} dx, \quad t > x > s.
\]

For \( \alpha = n \), formula (2.2) is the Cauchy integral formula,

\[
I_t^\alpha \psi (t) = \frac{1}{(n-1)!} \int_s^t (t-x)^{n-1}\psi (x) dx, \quad t > x > 0.
\]

**Theorem 2.2:** Let \( \alpha \in (n-1, n] \) and \( \psi(t), \phi(t) \) is any \( n \)-times differentiable function with respect to \( t \). Then,

1. \( T_t^\alpha (I_t^\alpha \psi) (t) = \psi(t) \),
2. \( I_t^\alpha (\mathcal{T}_t^\alpha \psi) (t) = \psi(t) - \frac{n-1}{(n-1)!} \sum_{n=0}^{\infty} \psi^{(n)}(s) (t-s)^{-\alpha} \)

Next, some outcomes associated with the fractional power series (FPS) in the sense of the time-CFD are gathered in order to formulate series expansion.

**Definition 2.3:** For \( \alpha \in (n-1, n] \), a power series of the form

\[
\sum_{m=0}^{\infty} f_m(r)(t-s)^{\mu m} = f_0(r) + f_1(r)(t-s)^{\nu} + \cdots, r \in \Omega, \ t \geq s \quad (2.3)
\]

is a multivariable FPS around \( t = s \), where the coefficients of the series, \( f_m(r) \), \( m = 0, 1, 2, \ldots \), are functions of \( r \).

**Theorem 2.3:** Assume that \( \psi(t) \) has a multivariable FPS a round \( t = s \),

\[
\psi(t) = \sum_{m=0}^{\infty} f_m(r)(t-s)^{\mu m}, \quad 0 \leq n-1 < \alpha \leq n, \ r \in \Omega,
\]

\[
s \leq t < s + R. \quad (2.4)
\]

If \( \psi(t) \) is \( m \)-times \( \alpha \)-differentiable with respect to \( t \) on \( \Omega \times (s, s+R) \), then the coefficients of the series in Eq. (2.4) are given as

\[
f_m(r) = \psi_t^{(m\alpha)}(s) \prod_{k=1}^{m} \Gamma(k\alpha - n + 1), \quad \Gamma(k\alpha+1), \quad (2.5)
\]

where \( \psi_t^{(m\alpha)} = \mathcal{T}_t^{\alpha} \cdot \mathcal{T}_t^{\alpha} \cdots \mathcal{T}_t^{\alpha} \psi \) and \( R \) is the radius of convergence of the series such that \( R = \min R_k \), in which \( R_k \) is the radius of convergence of the single variable FPS \( \sum_{n=0}^{\infty} f_n(s)(t-s)^{\mu n} \).

**Proof:** Let \( \psi(t) \) be a multivariable expansion that has a multivariable FPS expansion that is given in Eq. (2.3). Replacing \( t = s \) in
Eq. (2.4) hides all terms after the first one and hence determines the form of the first coefficient of the series
\[ f_0(r) = \psi(r, s). \] (2.6)

Apply the operator \( T_\epsilon \) once on Eq. (2.4) and then we will have the following expression:
\[ \psi_1(r, t) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n + 1)} f_0(r) + \frac{\Gamma(2\alpha + 1)}{\Gamma(2\alpha - n + 1)} f_1(r)(t - s)^n \]
\[ + \frac{\Gamma(3\alpha + 1)}{\Gamma(3\alpha - n + 1)} f_2(r)(t - s)^{2n} + \cdots. \] (2.7)

Now, by substituting \( t = s \) into Eq. (2.7), we can easily find the coefficient \( f_2(r) \) that takes the following form:
\[ f_2(r) = \psi_1^{(\alpha)}(r, s) - \frac{\Gamma(\alpha - n + 1)}{\Gamma(\alpha + 1)} f_0(r). \] (2.8)

Again, operating \( T_\epsilon \) twice on Eq. (2.4), we obtain
\[ \psi_2(r, t) = f_2(r) \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n + 1)} \frac{\Gamma(2\alpha + 1)}{\Gamma(2\alpha - n + 1)} f_0(r) + \frac{\Gamma(3\alpha + 1)}{\Gamma(3\alpha - n + 1)} \frac{\Gamma(4\alpha + 1)}{\Gamma(4\alpha - n + 1)} f_2(r)(t - s)^n \]
\[ + f_3(r) \frac{\Gamma(3\alpha + 1)}{\Gamma(3\alpha - n + 1)} \frac{\Gamma(4\alpha + 1)}{\Gamma(4\alpha - n + 1)} (t - s)^{2n} + \cdots. \] (2.9)

and replacing \( t = s \) in Eq. (2.9), we will get the form of the coefficient \( f_2(r) \), which has the form
\[ f_2(r) = \psi_1^{(2\alpha)}(r, s) - \frac{\Gamma(\alpha - n + 1)}{\Gamma(\alpha + 1)} f_0(r). \] (2.10)

The pattern is now clear; in fact, if we apply the operator \( T_\epsilon \) \( m \)-times on the Eq. (2.4) and replace \( t = s \) in the resulting equation, then we obtain the general form of \( f_m(r) \), which is
\[ f_m(r) = \psi_1^{(ma)}(r, s) \frac{\Gamma(\alpha - n + 1)}{\Gamma(ma - n + 1)} \frac{\Gamma(2\alpha - n + 1)}{\Gamma(2ma - n + 1)} \frac{\Gamma(3\alpha - n + 1)}{\Gamma(3ma - n + 1)} \frac{\Gamma(4\alpha - n + 1)}{\Gamma(4ma - n + 1)} \cdots \]
\[ \frac{\Gamma(\alpha - n + 1)}{\Gamma(\alpha + 1)} f_0(r), \quad m = 0, 1, 2, \ldots. \] (2.11)

and can be represented in the form that is given in Eq. (2.5).

**Remark 2.2:** (i) For a special case; if \( n = 1 \), then (2.11) becomes
\[ f_m(r) = \psi_1^{(ma)}(r, s) \frac{\phi(\alpha, m)}{\alpha^m(m!)} , \quad m = 0, 1, 2, \ldots. \] (2.12)

(ii) If we replace the formula of \( f_m(r) \) that is given in Eq. (2.11) into the series in Eq. (2.4), then we shall obtain a new form of the multivariable Taylor’s series formula in the sense of CFD as follows:
\[ \psi(r, t) = \sum_{n=0}^{\infty} \frac{\psi_1^{(ma)}(r, s)}{\phi(\alpha, m)} (t - s)^m, \quad 0 \leq n - 1 < \alpha \leq n, \]
\[ r \in \Omega, \quad s \leq t < s + R. \] (2.13)

where \( \phi(\alpha, m) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \frac{\Gamma(2\alpha + 1)}{\Gamma(2\alpha + 1)} \cdots \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)}, \quad m = 1, 2, \ldots, \)
and \( \phi(\alpha, 0) = 1. \)

(iii) If we set \( \alpha = 1 \) in Eq. (2.13), then the following usual multivariable Taylor’s series formula is obtained:
\[ \psi(r, t) = \sum_{m=0}^{\infty} \frac{\partial^m \psi(r, s)}{\partial t^m} \frac{(t - s)^m}{m!}, \quad r \in \Omega, \quad s \leq t < s + R. \] (2.14)

The new form of the Taylor’s series formula that is mentioned in Eq. (2.13) will be applied during this work.

### III. SOLITARY PATTERN SOLUTIONS FOR TFND-PDEs BY USING RFPS METHOD

In this section, we construct the solitary solutions for TFND-PDEs in the sense of the CFD by fractional power series method, and the recursion formulas for the coefficient’s computation are also derived by the RFPS technique.

From a mathematician viewpoint, the solutions of solitary are special solutions of some nonlinear equations of fractional or integer order. Examples of such equations are the Schrödinger, Korteweg-de Vries, Boussinesq, Klein-Gordon, \( B(\alpha, \beta, \gamma) \), and sine-Gordon equations. These equations have many applications in physics such as in plasmas, electro-magnetic, nonlinear optics, condensed matter, and fluid mechanics.

To achieve our aim in this work, we start with some illustration of the RFPS method by solving the target equations in this research analytically. Anyway, Eqs. (1.1)–(1.3) can be rewritten in a short form as follows:
\[ \psi_i(r, t) = N_i[\psi(r, t)], \quad 0 < \alpha \leq 1, \quad r \in \Omega, \quad t > s \geq 0, \] (3.1)
subject to
\[ \psi(r, s) = f(r), \]
\[ \psi^{(\alpha)}(r, s) = g(r), \] (3.2)
where \( \alpha \) refers to the order of the CFD and \( N_i \) is a linear or nonlinear differential operator with respect to the independent variable \( r \) that contains the unknown variables \( \psi, \psi', \psi'' \), and \( \psi''' \).

According to the RFPS method, we assume the solution of the initial value problem (IVP) (3.1) and (3.2) can be expressed in the following expansion:
\[ \psi(r, t) = \sum_{m=0}^{\infty} f_m(r) \frac{(t - s)^m}{\alpha^m(m!)} , \quad 0 < \alpha \leq 1, \quad r \in \Omega, \quad s \leq t < s + R. \] (3.3)

The RFPS solution of the IVP (3.1) and (3.2) can be approximated by truncating the series (3.3) up to the \( k \)-th terms. Therefore, an approximate numerical solution can be obtained from the truncated series. The following finite series that denoted by \( \psi_k(r, t) \) is the \( k \)-th–truncated series of \( \psi(r, t) \):
\[ \psi_k(r, t) = \sum_{m=0}^{k} f_m(r) \frac{(t - s)^m}{\alpha^m(m!)} , \quad 0 < \alpha \leq 1, \quad r \in \Omega, \quad s \leq t < s + R. \] (3.4)

From the conditions in Eq. (3.2) and the expansion (3.4), we conclude that \( f_k(r) = \psi(r, s) = f(r) \) and \( f_k(r) = \psi^{(\alpha)}(r, s) = g(r) \).
The leads to the algebraic equation
\[ \text{Res}_k = 0 \]
and can be obtained from the series by computing few terms only. Higher accuracy solution of the IVP (3.1) and (3.2) can be achieved by increasing the number of terms in the approximate solution in Eq. (3.4). As we shall see later, a solitary solution can be obtained from the series by computing few terms only.

**IV. TFN-PFS: THREE TYPES AND NUMERICAL SIMULATIONS**

In this section, we present the approximate solitary solutions of the TFN Boussinesq, TFN Klein-Gordon, and TFN B(2, 1, 1) PDEs in CDF sense by using the RFPS method. Numerical and graphical results are also given and conferred quantitatively to clarify the required solutions. MATHEMATICA 7 is used in our numerical computations. Our algorithm provides a solution to the problem in a fast-convergent series that grants an optimal solution.

**Problem 4.1:** Given the following TFN Boussinesq PDE:

\[ \psi_1^{2m}(r, t) - (\psi^2(r, t))_t - \psi_r(r, t) + (\psi(r, t)\psi_\psi(r, t))_r = 0, \]

\[ 0 < \alpha \leq 1, \quad r \in \mathbb{R}, \quad t > 0, \]

with the conditions

\[ \psi_1(r, 0) = -2(\omega^2 - 1)\sinh\left(\frac{1}{2r}\right), \]

\[ \psi_1^{(\alpha)}(r, 0) = \omega(\omega^2 - 1)\sinh(r), \]

where \( \omega \) is a constant.

Applying RFPS technique by considering that \( f_0(r) = -2(\omega^2 - 1)\sinh\left(\frac{1}{2r}\right) \) and \( f_1(r) = \omega(\omega^2 - 1)\sinh(r), \) gives the 1st-approximate of the RFPS solution of the IVP (4.1) and (4.2), which is the initial guess approximation of \( \psi(r, t) \) and given by

\[ \psi_1(r, t) = -2(\omega^2 - 1)\sinh\left(\frac{1}{2r}\right) + \omega(\omega^2 - 1)\sinh(r) \frac{r_\psi}{\alpha}. \]
\[ \text{Res}_k(r, t) = \frac{\partial^{3u}}{\partial t^{3u}} \left( \psi_k(r, t) - \frac{\partial^2}{\partial r^2} (\psi_0^2(r, t)) - \frac{\partial^2}{\partial r^2} (\psi_0(r, t)) + \frac{\partial^2}{\partial r^2} \left( \psi_k(r, t) \frac{\partial^2}{\partial r^2} \psi_k(r, t) \right) \right). \] (4.4)

In contrast as well, the 3rd-truncated FPS expansion about \( t = 0 \) of \( \psi(r, t) \), with the help of Eq. (3.5) and \( f_0(r) \) and \( f_1(r) \) forms, is given by

\[ \psi_k(r, t) = -2(\omega^2 - 1) \sinh^2 \left( \frac{1}{2} r \right) + \omega(\omega^2 - 1) \sinh \frac{t}{\alpha} + \sum_{n=2}^{\infty} f_n(r) \frac{\partial^n}{\partial t^n}(\psi_0(r, t)). \] (4.5)

If we substitute \( \psi_2(r, t) = -2(\omega^2 - 1) \sinh^2 \left( \frac{1}{2} r \right) + \omega(\omega^2 - 1) \sinh \frac{t}{\alpha} + f_2(r) \) (the 2nd approximation of the RFPS solution) into \( \text{Res}_2(r, t) = \frac{\partial^{3u}}{\partial t^{3u}} \left( \psi_2(r, t) - \frac{\partial^2}{\partial r^2} (\psi_2^2(r, t)) - \frac{\partial^2}{\partial r^2} (\psi_0(r, t)) + \frac{\partial^2}{\partial r^2} \left( \psi_2(r, t) \frac{\partial^2}{\partial r^2} \psi_0(r, t) \right) \) (the 2nd-residual function), we obtain \( f_2(r) \), which is the first unknown coefficient in Eq. (4.5) as follows:

\[ \text{Res}_2(r, t) = f_2(r) + \omega^2(\omega^2 - 1) \cosh r - \omega^3(\omega^2 - 1) \sinh \frac{t}{\alpha} + ((\omega^2 - 1) \cosh r f_2(r) + (\omega^2 - 1) \sinh r f'_2(r) + (1 - 2\omega^2) f''_2(r) + 2(1 - \omega^2) \sinh r f''_2(r) + \omega(\omega^2 - 1) f_2(r) \sinh r + 2 \cosh r f''_2(r) - 2 \cosh r f''_2(r) - \sinh x f''_2(r)) \frac{t^{3u}}{2\alpha^3} + ((f''_2(r)^2 - 2(f'_2(r))^2 + 2 f'_2(r)f''_2(r) + f_2(r)f''_2(r) - 2f''_2(r)) \frac{t^{3u}}{4\alpha^3}. \] (4.6)

Now, substituting \( t = 0 \) into Eq. (4.6) yields

\[ f_2(r) = -\omega^2(\omega^2 - 1) \cosh r. \] (4.7)

Thus, the 2nd-approximation of the RFPS solution of IVP (4.1) and (4.2) is given by

\[ \psi_2(r, t) = -2(\omega^2 - 1) \sinh^2 \left( \frac{1}{2} r \right) + \omega(\omega^2 - 1) \sinh \frac{t}{\alpha} - \omega^2(\omega^2 - 1) \cosh r \frac{t^{3u}}{2\alpha^3}. \] (4.8)

Similarly, to compute \( f_3(r) \) (the unknown second coefficient), substitute \( \psi_2(r, t) = \psi_2(r, t) + f_3(r) \frac{t^{3u}}{2\alpha^3} \) (the 3rd-approximate of the RFPS solution) into the 3rd-residual function \( \text{Res}_3(r, t) = \frac{\partial^{3u}}{\partial t^{3u}} \left( \psi_3(r, t) - \frac{\partial^2}{\partial r^2} (\psi_3^2(r, t)) - \frac{\partial^2}{\partial r^2} (\psi_0(r, t)) + \frac{\partial^2}{\partial r^2} \left( \psi_3(r, t) \frac{\partial^2}{\partial r^2} \psi_0(r, t) \right) \) to get

\[ \text{Res}_3(r, t) = (f_3(r) - \omega^2(\omega^2 - 1) \sinh r) \frac{t^{3u}}{\alpha} + \omega^4(\omega^2 - 1) f_3(r) \cosh r \frac{t^{3u}}{2\alpha^3} - (f''_3(r) - \omega^2 - 1)(2 \sinh r f'_3(r) + \cosh r f_3(r) - 2 \sinh r f''_3(r) - 2 f''_3(r) - \cosh r f''_3(r)) \frac{t^{3u}}{6\alpha^3} - \omega(\omega^2 - 1)(\sinh r f_3(r) + 2 \cosh r f'_3(r) - 2 \cosh r f''_3(r) - \sinh r f''_3(r)) \frac{t^{3u}}{6\alpha^3} + \omega^2(\omega^2 - 1)(\cosh r f_3(r) - 2 \sinh r f''_3(r) + 2 \sinh r f'_3(r) - \cosh r f''_3(r)) \frac{t^{3u}}{12\alpha^3} - 2(f''_3(r)^2 - (f''_3(r))^2 - 2 f'_3(r)f''_3(r) + f_3(r)(2 f''_3(r) + f''_3(r)) \frac{t^{3u}}{36\alpha^3}. \] (4.9)

Now, applying \( T^\alpha_r \) on Eq. (4.9) implies the time-CFD of order \( \alpha \) of \( \text{Res}_3(r, t) \) as follows:

\[ T^\alpha_r \text{Res}_3(r, t) = f_3(r) - \omega^2(\omega^2 - 1) \sinh r + \omega^4(\omega^2 - 1) \cosh r \frac{t^{3u}}{\alpha} - (f''_3(r) - \omega^2 - 1)(\cosh r f_3(r) + 2 \sinh r f'_3(r) - 2 f''_3(r)) \frac{t^{3u}}{2\alpha^3} \]

\[ - \omega(\omega^2 - 1)(\sinh r f_3(r) + 2 \cosh r f'_3(r) - 2 \cosh r f''_3(r) - \sinh r f''_3(r)) \frac{2t^{3u}}{3\alpha^3} + (\cosh r f_3(r) - 2 \sinh r f''_3(r) + 2 \sinh r f'_3(r) - \cosh r f''_3(r)) \frac{5t^{3u}}{12\alpha^3} - (2 f''_3(r)^2 - 2 f'_3(r)f''_3(r) - (f''_3(r))^2 + f_3(r)(2 f''_3(r) - f''_3(r)) \frac{t^{3u}}{6\alpha^3}. \] (4.10)
The $f_3(r)$ discretized form given in the following equation is obtained from Eq. (3.8) by assuming $k = 3$:

$$f_3(r) = \omega \left(\cos^2 - 1\right) \sinh r. \quad (4.11)$$

Consequently, the 3rd-approximate of the RFPS solution of IVP (4.1) and (4.2) is achieved and the previous results can be presented in the following form:

$$\psi_3(r, t) = -2(\omega^2 - 1) \sinh^{\frac{1}{2}} \left(\frac{1}{2} r\right) + \omega(\omega^2 - 1) \sinh r \frac{\mu}{F(1 + \alpha)}$$

$$- \omega^2 (\omega^2 - 1) \cosh r \frac{r^{2\alpha}}{2\alpha^2} + \omega^2 (\omega^2 - 1) \sinh r \frac{r^{2\alpha}}{6\alpha^2}. \quad (4.12)$$

As the earlier step, $f_4(r)$ discretized form given in the following equation is gained using the same process for $k = 4$

$$f_4(r) = -\omega^4 (\omega^2 - 1) \cosh r, \quad (4.13)$$

$$\psi_4(r, t) = (\omega^2 - 1) - (\omega^2 - 1) \cosh r$$

$$\times \left(1 + \omega \frac{t^{2\alpha}}{2\alpha^2} + \omega^4 \frac{t^{4\alpha}}{4\alpha^4} + \omega^8 \frac{t^{6\alpha}}{6\alpha^6} + \cdots \right)$$

$$+ (\omega^2 - 1) \sinh r \left(\omega \frac{t^{2\alpha}}{3\alpha^2} + \omega^4 \frac{t^{4\alpha}}{5\alpha^4} + \cdots \right). \quad (4.14)$$

we observe the pattern between the terms of the approximate solution in Eq. (4.6) and therefore we can predict the closed form of the solitary solution of IVP (4.1) and (4.2) in a form of combinations of cosine and sine hyperbolic functions coincides with the next form,

$$\psi(r, t) = -(\omega^2 - 1) \left[ \cosh r \cosh \left(\frac{\omega^2}{\alpha}\right) - \sinh r \sinh \left(\frac{\omega^2}{\alpha}\right) - 1 \right]. \quad (4.15)$$

On the other aspect as well, applying the same process deliberated in problem (4.1), the following solitary solution is gained by choosing the initial conditions as $\psi(x, 0) = 2(\omega^2 - 1) \cosh^6 \left(\frac{1}{2} x\right)$ and $u_0^\alpha(x, 0) = -\omega(\omega^2 - 1) \sinh x$ of the IVP (4.1) and (4.2):

$$\psi(r, t) = (\omega^2 - 1) \left[ \cosh r \cosh \left(\frac{\omega^2}{\alpha}\right) - \sinh r \sinh \left(\frac{\omega^2}{\alpha}\right) + 1 \right], \quad (4.16)$$

provided that $f_0(r), f_1(r), f_2(r),$ and $f_3(r)$ are given.

Now, calculate $f_5(r), f_6(r),$ and summarize $f_n(r), n = 1, 2, 3, 4$ to obtain

$$f_0(r) = -2(\omega^2 - 1) \sinh^{\frac{1}{2}} \left(\frac{1}{2} r\right) = -(\omega^2 - 1) \cosh r - 1,$$

$$f_1(r) = \omega (\omega^2 - 1) \cosh r,$$

$$f_2(r) = -\omega^2 (\omega^2 - 1) \cosh r, \quad (4.17)$$

$$f_3(r) = -\omega^2 (\omega^2 - 1) \cosh r,$$

$$f_4(r) = -\omega^4 (\omega^2 - 1) \cosh r. \quad (4.18)$$

Anyhow, the 6th-approximate of the RFPS solution of IVP (4.1) and (4.2), using the results of Eq. (4.14), can be truncated as follows:

$$\psi(r, t) = 2(\omega^2 - 1) \cosh \left(\frac{1}{2} (r - \omega t)\right). \quad (4.19)$$

**Remark 4.1:** It is interesting to point out that, when switching $\alpha$ by 1 in Eqs. (4.17) and (4.18), respectively, and doing some simplifications, we obtain the solutions achieved by the sine-cosine method, the Adomian decomposition method, and the variational iteration method, which have the following form:

$$\psi(r, t) = 2(\omega^2 - 1) \cosh \left(\frac{1}{2} (r - \omega t)\right).$$

**Problem 4.2:** The time-CFD Klein-Gordon PDE is given by

$$\psi_t^{(2\alpha)}(r, t) + a(\psi^2(r, t))_r + b(\psi^3(r, t))_{rr} = 0, \quad 0 < \alpha \leq 1,$$

$$r \in \mathbb{R}, \quad t < 0,$$

subject to

$$\psi(r, 0) = -4 \frac{\omega^2}{3\mu} \sinh^2 \left(\frac{1}{4} \sqrt{\frac{a}{b}} r\right), \quad (4.20)$$

$$\psi_{(r)}(r, 0) = 0 \quad (4.21)$$

where $a, b \in \mathbb{R}^+$ and $\omega$ is any constant.

According to the RFPS technique, $f_0(r) = -4 \frac{\omega^2}{3\mu} \sinh^2 \left(\frac{1}{4} \sqrt{\frac{a}{b}} r\right),$ $f_1(r) = \frac{a}{\sqrt{ab}} \sinh \left(\frac{\sqrt{a}}{\sqrt{b}} r\right)$ and so the initial guess approximation of $\psi(r, t)$ is $\psi_1(r, t) = -4 \frac{\omega^2}{3\mu} \sinh^2 \left(\frac{1}{4} \sqrt{\frac{a}{b}} r\right) + \frac{a}{\sqrt{ab}} \sinh \left(\frac{\sqrt{a}}{\sqrt{b}} r\right) \frac{\sqrt{a}}{\sqrt{b}},$ the 4th-residual function $\text{Res}_4(r, t) = \frac{\alpha}{4\gamma} \psi_4(r, t) - a \frac{\alpha^2}{4\gamma} \psi_4^2(r, t) + \frac{b}{\alpha^2} \psi_4(r, t)$, and the space-differential operator $N_r[\psi(r, t)] = -a
\( (\psi^2(r, t), \tau + b(\psi^2(r, t))_{,rr}, \) in which are used through the computations and steps in Sec. III; the unknown coefficients \( f_\alpha(r), \) \( n = 2, 3, 4, 5, 6 \) are obtained in the following forms:

\[
f_2(r) = -\frac{\omega^4}{6b} \cosh \left( \sqrt{\frac{a}{b}} \frac{r}{2} \right),
\]

\[
f_3(r) = \frac{\omega^5}{12} \sqrt{\frac{a}{b}} \sinh \left( \sqrt{\frac{a}{b}} \frac{r}{2} \right),
\]

\[
f_4(r) = -\frac{a \omega^6}{24b^2} \cosh \left( \sqrt{\frac{a}{b}} \frac{r}{2} \right),
\]

\[
(4.20)
\]

Therefore, the solitary solution of IVP (4.20) and (4.21) can be written in terms of hyperbolic functions as follows:

\[
\psi_s(r, t) = -\frac{2a^2}{3a} \left[ \cosh \left( \frac{\sqrt{a} r}{2} \right) \cosh \left( \frac{\sqrt{a} t^\alpha}{b} \right) \right] - \sinh \left( \frac{\sqrt{a} r}{2} \right) \sinh \left( \frac{\sqrt{a} t^\alpha}{b} \right) - 1.
\]

(4.23)

Remark 4.2: It is interesting to point out that, replacing \( \alpha \) by 1 in Eqs. (4.25) and (4.26), respectively, and using some identities, the solitary solutions of Eqs. (4.20) and (4.21) are obtained as follows:

\[
\psi(r, t) = -\frac{4a^2}{3a} \sinh^2 \left( \frac{\sqrt{a} t}{b} (r - \alpha t) \right),
\]

(4.27)

which are precisely the solutions achieved by the sine-cosine method, the Adomian decomposition method \( 3 \) and the variational iteration method.\( 5 \)

The solution geometric behavior of Eqs. (4.20) and (4.21) is investigated subsequently by sketching in 3D the figure of the 10th-approximate of the RFPS solution of Eq. (4.24) using different categories. Anyhow, the scenarios in Figs. (a)–(c) are to plot 10th-approximate of the RFPS solution on the domain \([-10, 10] \times [0, 1], \psi_\alpha(r, t), \text{ when } \alpha = 0.5, \alpha = 0.75 \text{ and } \alpha = 1, \) respectively. The exact solution given by Eq. (4.25) for \( \alpha = 1 \) is shown in Fig. 1(d).

It is clear in Fig. 1 that by changing the fractional parameter \( \alpha, \) we can increase or decrease the amplitudes of the solitary pattern solutions.
Referring to Fig. 1 in Ref. 30, which represents the solitary solution of the time-fractional Klein-Gordon PDE in the sense of Caputo’s fractional derivative, we see that the surface graphs simulate the surface graphs of Eq. (4.20) that is displayed in Fig. 1.

**Problem 4.3:** The time-CFD B(2, 1, 1)-type PDE is given by

\[
\psi_t^{(2\alpha)}(r, t) + a(\psi^2(r, t))_r - b(\psi(r, t)\psi_r(r, t))_r = 0, \quad 0 < \alpha \leq 1, \quad r \in \mathbb{R}, \quad t > 0, \quad (4.28)
\]

with the initial conditions

\[
\psi(r, 0) = \frac{2\omega^2}{a} \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} r \right),
\]

where \(a, b \in \mathbb{R}^+\) and \(\omega\) is any constant.

According to the procedure of the RFPS method, the first and second coefficients of the series in Eq. (3.5) are \(f_0(r) = \frac{2\omega^2}{a} \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} r \right)\) and \(f_1(r) = -\frac{\omega^3}{\sqrt{ab}} \sinh \left( \sqrt{\frac{a}{b}} r \right)\), the initial guess approximation of the RFPS solution of IVP (4.28) and (4.29) is \(\psi_1(r, t) = \frac{2\omega^2}{a} \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} r \right) - \frac{\omega^3}{\sqrt{ab}} \sinh \left( \sqrt{\frac{a}{b}} r \right) \frac{t}{\alpha}\), the kth-residual function of Eq. (4.28) is \(\text{Res}_k(r, t) = \frac{\omega^3}{\sqrt{ab}} \left( \psi_k(r, t) + a \frac{\omega^2}{a} \psi_1^2(r, t) \right) - b \frac{\omega^3}{\sqrt{ab}} \psi_1(r, t)\), and the space-differential operator is \(N_r[\psi(r, t)] = a(\psi^2)(r, t) + b(\psi(r, t)\psi_r(r, t))_r\). Therefore, the...
first six approximations of the RFPS solution of the IVP (4.28) and (4.29) are given by

\[
\psi(0, t) = \frac{2a^2}{a^2} \sinh \left( \frac{1}{2} \sqrt{\frac{a}{b}} \right) \left( 1 - \frac{a^2}{b} \right) \left( \cosh \left( \sqrt{\frac{a}{b}} t \right) - 1 \right),
\]

\[
\psi_1(0, t) = \psi(0, t) - \frac{a^3}{\sqrt{ab}} \sin \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{\alpha^3},
\]

\[
\psi_2(0, t) = \psi_1(0, t) - \frac{a^4}{b} \cosh \left( \sqrt{\frac{a}{b}} t \right) \frac{t^\omega}{2!a^{2\omega}},
\]

\[
\psi_3(0, t) = \psi_2(0, t) - \frac{a^5}{b^{3/2}} \sin \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{3!a^{3\omega}}, (4.30)
\]

\[
\psi_4(0, t) = \psi_3(0, t) + \frac{a^6}{b} \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{4!a^{4\omega}},
\]

\[
\psi_5(0, t) = \psi_4(0, t) - \frac{a^7}{b^{5/2}} \sin \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{5!a^{5\omega}},
\]

\[
\psi_6(0, t) = \psi_5(0, t) + \frac{a^8}{b^3} \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{6!a^{6\omega}}.
\]

Thus, the full terms of the 6th-approximate of the RFPS solution are as follows:

\[
\psi_6(0, t) = \frac{a^5}{a^2} \left( \cosh \left( \sqrt{\frac{a}{b}} \right) - 1 \right) - \omega_3 \sin \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{\alpha^3} + \omega_4 \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{2!a^{2\omega}} - \omega_5 \sin \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{3!a^{3\omega}} + \omega_6 \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{4!a^{4\omega}} - \omega_7 \sin \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{5!a^{5\omega}} + \omega_8 \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{6!a^{6\omega}}. (4.31)
\]

In fact, if we separate and collect the terms of the solution in Eq. (4.31) for two sets contain \((2n)\alpha\) and \((2n + 1)\alpha, n = 0, 1, 3, \ldots\) terms, then we can discover the pattern in the terms of the series solution. Hence, the RFPS solution has the following general form:

\[
\psi(0, t) = \frac{a^5}{a^2} \left[ \cosh \left( \sqrt{\frac{a}{b}} \right) \left( 1 + \frac{a^2}{b} \frac{t^\omega}{2!a^{2\omega}} + \frac{a^4}{b^2} \frac{t^\omega}{4!a^{4\omega}} + \cdots - 1 \right) \right] - \omega_3 \sin \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{\alpha^3} + \omega_4 \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{2!a^{2\omega}} - \omega_5 \sin \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{3!a^{3\omega}} + \omega_6 \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{4!a^{4\omega}} - \omega_7 \sin \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{5!a^{5\omega}} + \omega_8 \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{6!a^{6\omega}} + \cdots. (4.32)
\]

So, the closed form of the solitary solution of the IVP (4.28) and (4.29) has the following expression:

\[
\psi(r, t) = \frac{a^5}{a^2} \left[ \cosh \left( \sqrt{\frac{a}{b}} \right) \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{\alpha^3} - \sin \left( \frac{a}{\sqrt{b}} \right) \sin \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{\alpha^3} - 1 \right]. (4.33)
\]

On the other aspect as well, if we repeat the same process deliberated in this example by choosing the initial conditions as \(\psi(r, 0) = -\frac{2a^2}{b} \cosh \left( \frac{a}{\sqrt{b}} \right) \cos \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{\alpha^3} \) instead of initial conditions in Eq. (4.29), then we gain the following solitary solution:

\[
\psi(r, t) = -\frac{a^5}{a^2} \left[ \cosh \left( \sqrt{\frac{a}{b}} \right) \cosh \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{\alpha^3} - \sin \left( \frac{a}{\sqrt{b}} \right) \sin \left( \sqrt{\frac{a}{b}} \right) \frac{t^\omega}{\alpha^3} + 1 \right]. (4.34)
\]

**Remark 4.3:** If we choose \(\alpha = 1\) for Eqs. (4.33) and (4.34), then we acquire the subsequent solitary wave solutions

\[
\psi(r, t) = \frac{2a^2}{a^2} \sinh \left( \frac{1}{\sqrt{b}} \right) \cos \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{\alpha^3},
\]

\[
\psi(r, t) = -\frac{a^5}{a^2} \cosh \left( \frac{a}{\sqrt{b}} \right) \cos \left( \frac{a}{\sqrt{b}} \right) \frac{t^\omega}{\alpha^3},
\]

which are exactly the solutions achieved by the sine-cosine method, the Adomian decomposition method, and the variational iteration method.}

**TABLE I.** Absolute error of the 10th-approximate of the RFPS solution for the IVP (4.28) and (4.29),

| \(r\) | \(t\) \(t\) | \(\alpha = 0.5\) \(\alpha = 0.75\) \(\alpha = 1\) |
|---|---|---|---|
| 0.25 | 5.5511 \times 10^{-17} | 2.3592 \times 10^{-16} | 1.3877 \times 10^{-17} |
| 0.50 | 5.5511 \times 10^{-17} | 2.3592 \times 10^{-16} | 1.3877 \times 10^{-17} |
| 0.75 | 5.5511 \times 10^{-17} | 2.3592 \times 10^{-16} | 1.3877 \times 10^{-17} |
| 1 | 5.5511 \times 10^{-17} | 2.3592 \times 10^{-16} | 1.3877 \times 10^{-17} |

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In order to determine the accuracy of the proposed method, Table I displays the absolute error |υ(r, t) − υh(r, t)| of the 10th approximate of the RFPS solution for the IVP (4.28) and (4.29). For numerical results, the following values, for parameters, are considered: a = b = ω = 1.

Table I shows the numerical approximation of the solitary solution using the RFPS method; however, t is obvious from Table I, the approximate solutions are very close to the exact solutions for any chosen values of r and t , while accuracy is high by calculating a few terms of the series solution.

**V. CONCLUSIONS**

Given the plentiful advantages of CFD definition, many researchers have studied the appropriateness of using it as an alternative to other fractional derivatives in modeling many natural phenomena. In this work, we aim to achieve three goals: first, preparing a new form of the Taylor’s series formula in the sense of CFD and employing it in the residual power series method (RPSM) to introduce solutions for the PDEs; second, presenting solitary pattern solutions for the TFND-PDEs in the sense of CFD by using RPSM; finally, comparing the behavior of the resulting solutions of the targeted equations with the solutions prior to those equations in the sense of Caputo fractional derivative. All objectives have been achieved and the results indicate that the RPSM is an efficient and easy tool in finding exact solitary pattern solutions for the TFND Boussinesq, TFND Klein–Gordon, and TFND B(2,1,1) PDEs. The calculations used to find a CFD are much easier than finding the Caputo fractional derivative. The surface graphs of the solutions of the target equations in the sense of CFD simulate the surface graphs of the solutions in the sense of the Caputo fractional derivatives. Therefore, the CFD definition can be used as an alternative to other fractional derivatives in the remodeling of other differential equations. How to apply RFPS method for finding exact and numerical solutions of conformable fractional PDEs that are coming from other sciences such as conformable fractional time-varying descriptor equations still need further research.

**REFERENCES**