**Dr. Mahmoud M. Al-Husari**

# Signals and Systems Lecture Notes

*This set of lecture notes are never to be considered as a substitute to the textbook recommended by the lecturer.*

# Contents

# Preface

I have written this set of lecture notes to my students taking the signals and system course taught at the University of Jordan. It covers a one-semester course in the basics of signal and system analysis during the junior year. I was motivated by the fact that no textbook on this subject covers the exact course content described by the departments catalogue.

This set of lecture notes grew out of class notes I have written in this area for many years. My class notes were collected from many sources covering this area. I don't claim that the material presented here is a genuine contribution in the area of signal and systems analysis. It is rather a collection of different topics, examples and illustrations from different books, other lecture notes, and web sites. It is organized in a manner to make it easier for my students to sail smoothly between chapters in the same order required by the course content. From each garden I simply tried to pick the best rose.

Authors writing textbooks covering the topic of signal and systems analysis are divided into two major campaigns. The first campaign prefers in their treatment of the topic to give comprehensive treatment of the continuous-time signals and systems in the first part of their textbooks. In the second part, they extend the results to cover discrete-time signals and systems. On the other hand, the second campaign of authors covers both continuous-time and discrete-time signals and systems together in a parallel approach of their treatment of the topic. In my experience, I have found that the latter approach confuses the students. However, I stand between the two campaigns. My approach is to introduce discrete-time signals and systems at an early stage in the course without giving comprehensive treatment of the discrete-time. Many developments of the theory and analysis of signals and systems are easier to understand in discrete-time first. Whenever this is the case the discrete-time system is presented first.

The notes begins with mathematical representations of signals and systems, in both continuous and discrete time. Chapter 1 is an introduction to the general concepts involved in signal and systems analysis. Chapter 2 covers signals, while Chapter 3 is devoted to systems.

To this point, the time-domain description of signals and systems for both continuous and discrete time is thoroughly covered. Next, we turn our attention to the frequency-domain descriptions of continuous-time signals and systems. In Chapter 4, the Fourier series representation of periodic signals and their properties are presented. Chapter 5 begins with the development of the Fourier transform and its properties are discussed. Applications of the Fourier transform in areas such as signal filtering, amplitude modulation, and sampling are considered in Chapter 6.

I have tried hard to make this set of lecture notes error free. I encourage students to draw my attention to any mistakes detected. I welcome any comments and suggestions. I wish to thank Dr. Ahmad Mustafa with whom I sometimes teach the course. His review of this set of notes and his valuable comments are much appreciated, it helped to make the set of notes better.

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Chapter 1

# Introduction

## 1.1 Signals and Systems Defined

The course objective is to present a mathematical study of *signals* and *systems*. Why study Signals and Systems? The concepts of signals and systems arise in virtually all areas of technology, including electrical circuits, communication devices, signal processing devices, control systems, and biomedical devices.

Since this course is about signals and systems, the first question to answer, What are they? What is a signal? A vague and not mathematically rigor definition is simply: A signal is something that contains information. The traffic light signal shown in Figure 1.1 provide us with information. If the lights are red you have to stop, on the other hand, if the lights are green you can proceed. Figure 1.2 illustrates more examples of signals providing information one way or another. *Formal Definition*: A signal is defined as a function of one or more variables which conveys information on the nature of a physical phenomenon. In other words, any time-varying physical phenomenon which is intended to convey information is a signal.

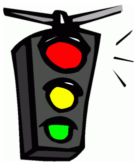


**Figure 1.1:** Traffic light signal.

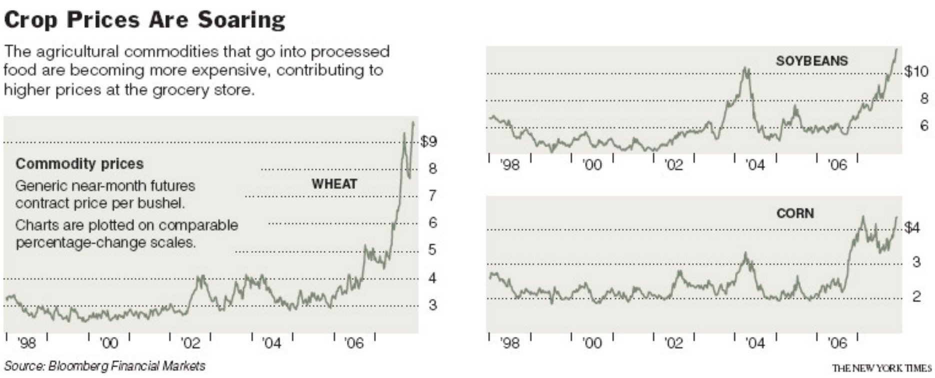



**Figure 1.2:** Examples of Signals.

Signals are processed or operated on by systems. What is a system? *Formal Definition*: A system is defined as an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals. When one or more excitation signals are applied at one or more system inputs, the system produces one or more response signals at its outputs. Throughout my lecture notes I will simply refer to the excitation signals applied at the input as the input signal. The response signal at the output will be referred to as the output signal. Figure 1.3 shows a diagram of a single-input, single-output system.

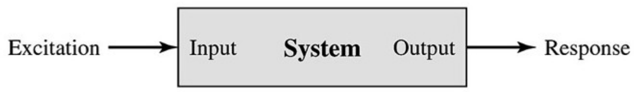



**Figure 1.3:** Block diagram of a simple system [1].

Systems with more than one input and more than one output are called MIMO (Multi-Input Multi-Output). Figure 1.4 depicts the basic elements of a communication system, namely, transmitter, channel, and receiver. The transmitter, channel, and receiver may be viewed as a system with associated signals of its own. The input signal (information signal) could be a speech signal for example. The transmitter converts the input signal into a form suitable for transmission over the channel. Due to presence of noise in the communication system, the received signal is a corrupted version of the transmitted signal. The receiver operates on the received signal so as to produce an estimate of the original input signal.

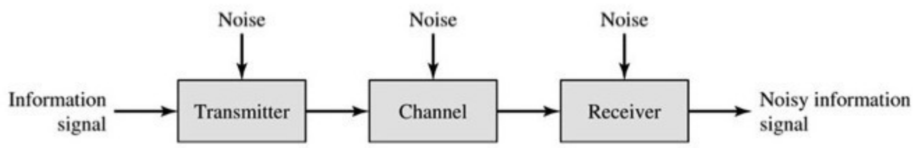



**Figure 1.4:** A communication system [1].

## 1.2 Types of Signals and Systems

Signals and systems are classified into two main types:

- *Continuous-time.*
- *Discrete-time.*

These two types can be divided into classes, as will be seen in Chapter 2, that is convenient in studying signals and systems analysis.

### 1.2.1 Signals

A *continuous-time* (CT) signal is one which is defined at every instant of time over some time interval. They are functions of a continuous time variable. We often refer to a CT signal as $x(t)$. The independent variable is time $t$ and can have any real value, the function $x(t)$ is called a CT function because it is defined on a continuum of points in time. It is very important to observe

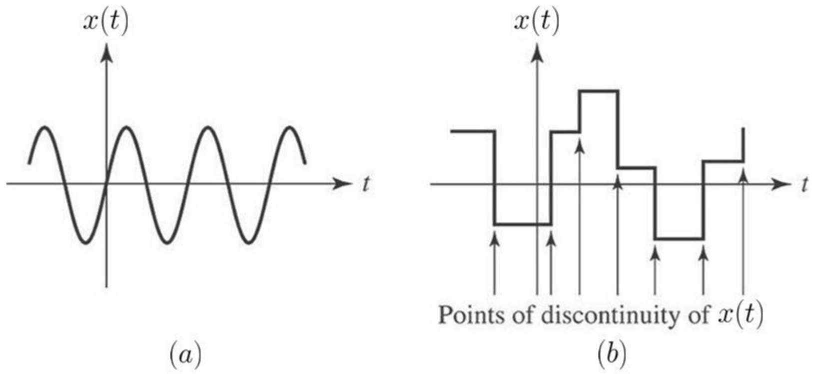



**Figure 1.5:** Example of CT signal.

here that Figure 1.5(b) illustrates a discontinuous function. At discontinuity, the limit of the function value as we approach the discontinuity from above is not the same as the limit as we approach the same point from below. Stated mathematically, if the time $t = t_0$ is a point of discontinuity of a function $x(t)$, then

$$\lim_{\epsilon \to 0} x(t + \epsilon) \neq \lim_{\epsilon \to 0} x(t - \epsilon)$$

However, the two functions shown in Figure 1.5 are continuous-time functions because their values are defined on a continuum of times $t$ ($t \in \mathcal{R}$), where $\mathcal{R}$ is the set of all real values. Therefore, the terms continuous and continuous time mean different things. A CT function is defined on a continuum of times, but is not necessarily continuous at every point in time. A discrete-time (DT) signal is one which is defined only at discrete points in time and not between them. The independent variable takes only a discrete set of values. We often refer to DT signal as $x[n]$, $n$ here belongs to the set of all integers $\mathcal{Z}$ ($n \in \mathcal{Z}$) i.e. $n = 0, \pm 1, \pm 2, \cdots$, (Figure 1.6). However, the amplitude is continuous and may take a continuum of values. A signal whose amplitude can take on any value in a continuous range is called an *analog signal*. A *digital signal*, on the other hand, is one whose amplitude can take on only a finite number of values. The terms continuous-time and discrete-time qualify the nature of the

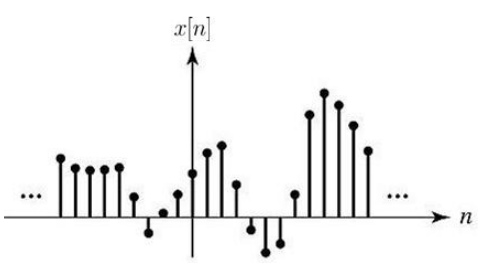



**Figure 1.6:** Example of a DT function.

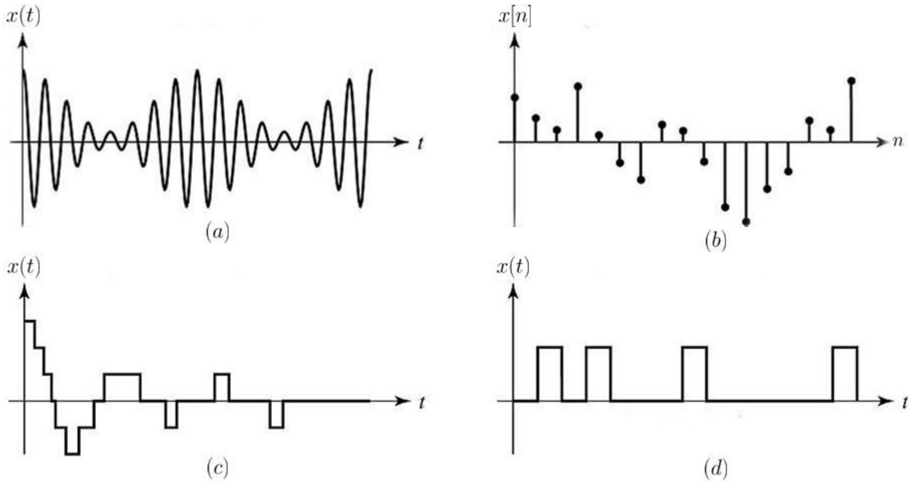



**Figure 1.7:** Examples of signals: (a) analog, continuous-time (b) analog, discrete-time (c) and (d) digital, continuous-time.

a signal along the time (horizontal) axis. The terms analog and digital, on the other hand, qualify the nature of the signal amplitude (vertical axis). Figure 1.7 shows examples of various types of signals.

### 1.2.2 Systems

A CT system transforms a continuous time input signal into CT outputs. Similarly a DT system transforms a discrete time input signal to a DT output signal as shown in Figure 1.8. In Engineering disciplines, problems that often arise are of the form

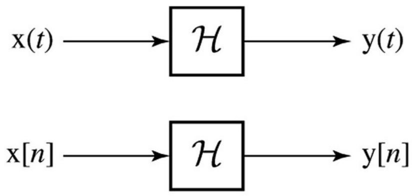



**Figure 1.8:** CT and DT system block diagram.

- Analysis problems
- Synthesis problems

In Analysis problems one is usually presented with a specific system and is interested in characterizing it in detail to understand how it will respond to various inputs. On the other hand, Synthesis problems requires designing systems to process signals in a particular way to achieve desired outputs. Our main focus in this course are analysis problems.

# Chapter 2

# Signals Representations

## 2.1 Classification of CT and DT Signals

### 2.1.1 Periodic and non-periodic Signals

A periodic function is one which has been repeating an exact pattern for an infinite period of time and will continue to repeat that exact pattern for an infinite time. That is, a periodic function $x(t)$ is one for which

$$x(t) = x(t + nT) \tag{2.1}$$

for any integer value of $n$, where $T > 0$ is the period of the function and $-\infty < t < \infty$. The signal repeats itself every $T$ *sec.* Of course, it also repeats every $2T, 3T$ and $nT$. Therefore, $2T, 3T$ and $nT$ are all periods of the function because the function repeats over any of those intervals. The *minimum positive* interval over which a function repeats itself is called the *fundamental period* $T_0$, (Figure 2.1). $T_0$ is the smallest value that satisfies the condition

$$x(t) = x(t + T_0) \tag{2.2}$$

The *fundamental frequency* $f_0$ of a periodic function is the reciprocal of the fundamental period $f_0 = \frac{1}{T_0}$. It is measured in *Hertz* and is the number of cycles (periods) per second. The fundamental *angular frequency* $\omega_0$ measured in radians per second is

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0. \tag{2.3}$$

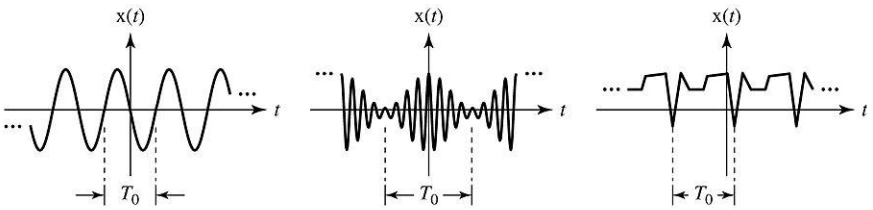



**Figure 2.1:** Example of periodic CT function with fundamental period.

A signal that does not satisfy the condition in (2.1) is said to be *aperiodic* or *non-periodic.*

**Example 2.1**

With respect to the signal shown in Figure 2.2 determine the fundamental frequency and the fundamental angular frequency.

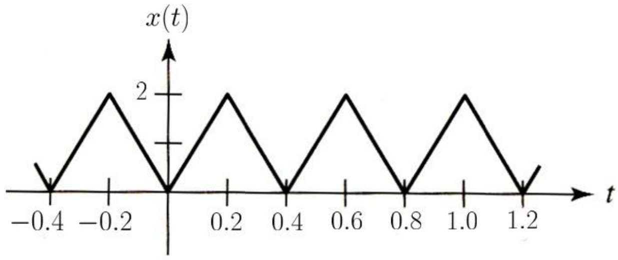



**Figure 2.2:** A triangular periodic signal.

■ **Solution** It is clear that the fundamental period $T_0 = 0.4sec$. Thus,

$$\begin{aligned} f_0 &= \frac{1}{0.4} = 2.5\text{Hz} \\ \omega_0 &= 2\pi f_0 = 5\pi\ rad/sec. \end{aligned}$$

It repeats itself 2.5 cycles/sec, which can be clearly seen in Figure 2.2. ■

**Example 2.2**

A real valued sinusoidal signal $x(t)$ can be expressed mathematically by

$$x(t) = A\sin(\omega_0 t + \phi) \tag{2.4}$$

Show that $x(t)$ is periodic.

■ **Solution** For $x(t)$ to be periodic it must satisfy the condition $x(t) = x(t+T_0)$, thus

$$\begin{aligned} x(t+T_0) &= A\sin(\omega_0(t+T_0)+\phi) \\ &= A\sin(\omega_0 t + \phi + \omega_0 T_0) \end{aligned}$$

Recall that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, therefore

$$x(t+T_0) = A\left[\sin(\omega_0 t+\phi)\cos\omega_0 T_0 + \cos(\omega_0 t+\phi)\sin\omega_0 T_0\right] \tag{2.5}$$

Substituting the fundamental period $T_0 = \frac{2\pi}{\omega_0}$ in (2.5) yields

$$\begin{aligned} x(t+T_0) &= A\left[\sin(\omega_0 t+\phi)\cos 2\pi + \cos(\omega_0 t+\phi)\sin 2\pi\right] \\ &= A\sin(\omega_0 t+\phi) \\ &= x(t) \quad ■ \end{aligned}$$

An important question for signal analysis is whether or not the sum of two periodic signals is periodic. Suppose that $x_1(t)$ and $x_2(t)$ are periodic signals with fundamental periods $T_1$ and $T_2$, respectively. Then is the sum $x_1(t)+x_2(t)$ periodic; that is, is there a positive number $T$ such that

$$x_1(t+T) + x_2(t+T) = x_1(t) + x_2(t) \quad \text{for all } t? \tag{2.6}$$

It turns out that (2.6) is satisfied if and only if the ratio $\frac{T_1}{T_2}$ can be written as the ratio $\frac{k}{l}$ of two integers $k$ and $l$. This can be shown by noting that if $\frac{T_1}{T_2} = \frac{k}{l}$, then $lT_1 = kT_2$ and since $k$ and $l$ are integers $x_1(t) + x_2(t)$ are periodic with period $lT_1$. Thus the expression (2.6) follows with $T = lT_1$. In addition, if $k$ and $l$ are co-prime (i.e. $k$ and $l$ have no common integer factors other than 1,) then $T = lT_1$ is the fundamental period of the sum $x_1(t) + x_2(t)$. In words, if a time can be found inside which both functions have an integer number of periods, then the sum will repeat with that period..

Let $x_1(t) = cos\left(\frac{\pi t}{2}\right)$ and $x_2(t) = cos\left(\frac{\pi t}{3}\right)$, determine if $x_1(t) + x_2(t)$ is periodic. **Example 2.3**

■ **Solution** $x_1(t)$ and $x_2(t)$ are periodic with the fundamental periods $T_1 = 4$ (since $\omega_1 = \frac{\pi}{2} = \frac{2\pi}{T_1} \Longrightarrow T_1 = 4$) and similarly $T_2 = 6$. Now

$$\frac{T_1}{T_2} = \frac{4}{6} = \frac{2}{3}$$

then with $k = 2$ and $l = 3$, it follows that the sum $x_1(t) + x_2(t)$ is periodic with fundamental period $T = lT_1 = (3)(4) = 12$ sec. The 12 seconds interval is the shortest time in which both signals have an integer number of periods. This time is then the fundamental period of the overall function. There are three fundamental periods of the first function and two fundamental periods of the second function in that time. ■

### 2.1.2 Deterministic and Random Signals

*Deterministic Signals* are signals who are completely defined for any instant of time, there is no uncertainty with respect to their value at any point of time. They can also be described mathematically, at least approximately. Let a function be defined as

$$\text{tri}(t) = \begin{cases} 1 - |t|, & -1 < t < 1 \\ 0, & \textit{otherwise} \end{cases}$$

for example. It is called the unit triangle function since its height and area are both one as shown in Figure 2.3. Clearly, this function is well defined mathematically.

A *random* signal is one whose values cannot be predicted exactly and cannot be described by any exact mathematical function, they can be approximately

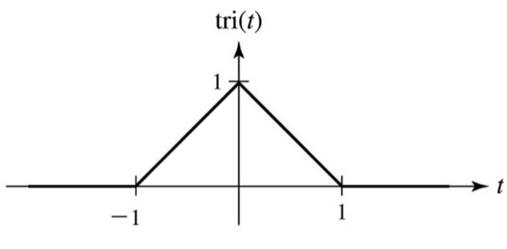



**Figure 2.3:** Example of deterministic signal.

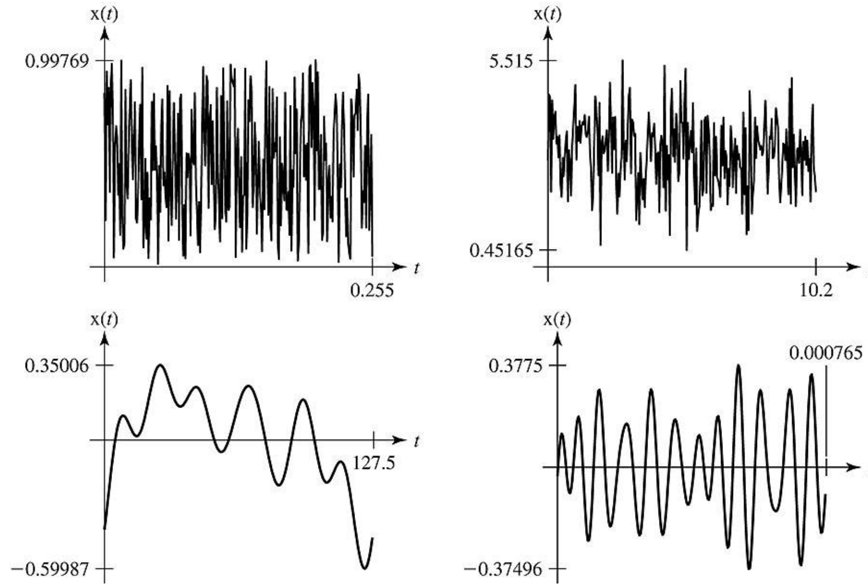



**Figure 2.4:** Examples of Noise.

described. A common name for random signals is noise, Figure 2.4 illustrates four different random continuous-time signals.

### 2.1.3 Signal Energy and Power

#### Size of a Signal

The size of any entity is a number that indicates the largeness or strength of that entity. Generally speaking, the signal amplitude varies with time. How can a signal as a one shown in Figure 2.3 for example, that exists over a certain time interval with varying amplitude be measured by one number that will indicate the signal size or signal strength? One must not consider only signal amplitude but also the duration. If for instance one wants to measure the size of a human by a single number one must not only consider his height but also his width. If we make a simplifying assumption that the shape of a person is a cylinder of variable radius $r$ (which varies with height $h$) then a reasonable measure of a human size of height $H$ is his volume given by

$$V = \pi \int_0^H r^2(h)dh$$

Arguing in this manner, we may consider the area under a signal as a possible measure of its size, because it takes account of not only the amplitude but also the duration. However this will be a defective measure because it could be a large signal, yet its positive and negative areas could cancel each other, indicating a signal of small size. This difficulty can be corrected by defining the signal size as the area under the square of the signal, which is always positive.

We call this measure the *Signal Energy* $E_\infty$, defined for a real signal $x(t)$ as

$$E_\infty = \int_{-\infty}^{\infty} x^2(t)\,dt \tag{2.7}$$

This can be generalized to a complex valued signal as

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2\,dt \tag{2.8}$$

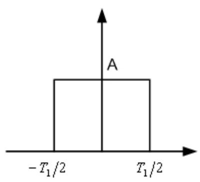
(Note for complex signals $|x(t)|^2 = x(t)x^*(t)$ where $x^*(t)$ is the complex conjugate of $x(t)$). Signal energy for a DT signal is defined in an analogous way as

$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{2.9}$$

**Example 2.4**

Find the signal energy of

$$x(t) = \begin{cases} A, & |t| < T_1/2 \\ 0, & otherwise \end{cases}$$




*Plotting $x(t)$ is helpful, as sometimes you do not need to determine the integral. You can find the area under the square of the signal from the graph instead.*

■ **Solution** From the definition in (2.7)

$$E_\infty = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-T_1/2}^{T_1/2} A^2 dt$$
$$= \left[A^2 t\right]_{-T_1/2}^{T_1/2} = A^2 T_1. \quad ■$$

For many signals encountered in signal and system analysis, neither the integral in

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

nor the summation

$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

converge because the signal energy is infinite. The signal energy must be finite for it to be a meaningful measure of the signal size. This usually occurs because the signal in not time-limited (*Time limited* means the signal is nonzero over only a finite time.) An example of a CT signal with infinite energy would be a sinusoidal signal

$$x(t) = A\cos(2\pi f_0 t).$$

For signals of this type, it is usually more convenient to deal with the average signal power of the signal instead of the signal energy. The average signal power of a CT signal is defined by

$$P_\infty = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\,dt \tag{2.10}$$

Some references use the definition

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt \tag{2.11}$$

Note that the integral in (2.10) is the signal energy of the signal over a time $T$, and is then divided by $T$ yielding the average signal power over time $T$. Then as $T$ approached infinity, this average signal power becomes the average signal power over all time. Observe also that the signal power $P_\infty$ is the time average (mean) of the signal amplitude squared, that is, the *mean-squared* value of $x(t)$. Indeed, the square root of $P_\infty$ is the familiar *rms*(root mean square = $\sqrt{P_\infty}$) value of $x(t)$.

For DT signals the definition of signal power is

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N}\sum_{n=-N}^{N-1} |x[n]|^2 \tag{2.12}$$

which is the average signal power over all discrete time.

For periodic signals, the average signal power calculation may be simpler. The average value of any periodic function is the average over any period. Therefore, since the square of a periodic function is also periodic, for periodic CT signals

$$P_\infty = \frac{1}{T}\int_{T} |x(t)|^2\,dt \tag{2.13}$$

where the notation $\int_T$ means the integration over one period ($T$ can be any period but one usually chooses the fundamental period).

**Example 2.5** Find the signal power of

$$x(t) = A\cos(\omega_0 t + \phi)$$

■ **Solution** From the definition of signal power for a periodic signal in (2.13),

$$P_\infty = \frac{1}{T}\int_{T} |A\cos(\omega_0 t + \phi)|^2\,dt = \frac{A^2}{T_0}\int_{-T_0/2}^{T_0/2} \cos^2\left(\frac{2\pi}{T_0}t + \phi\right)dt \tag{2.14}$$

Using the trigonometric identity

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

in (2.14) we get

$$P_\infty = \frac{A^2}{2T_0}\int_{-T_0/2}^{T_0/2}\left[1 + \cos\left(\frac{4\pi}{T_0}t + 2\phi\right)\right]dt \tag{2.15}$$

$$= \frac{A^2}{2T_0}\int_{-T_0/2}^{T_0/2} dt\ + \underbrace{\frac{A^2}{2T_0}\int_{-T_0/2}^{T_0/2}\cos\left(\frac{4\pi}{T_0}t + 2\phi\right)dt}_{=0} \tag{2.16}$$

the second integral on the right hand side of (2.16) is zero because it is the integral of a sinusoid over exactly two fundamental periods. Therefore, the signal power is $P_\infty = \frac{A^2}{2}$. Notice that this result is independent of the phase $\phi$ and the angular frequency $\omega_0$. It depends only on the amplitude $A$. ■

**Example 2.6**

Find the power of the signal shown in Figure 2.5.

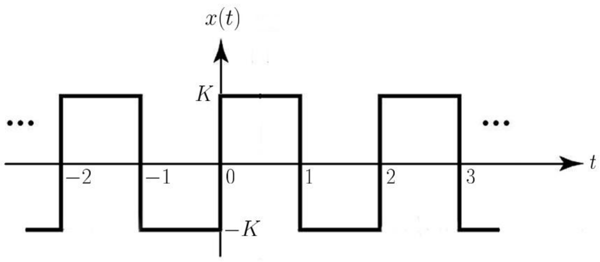



**Figure 2.5:** A periodic pulse.

■ **Solution** From the definition of signal power for a periodic signal

$$P_\infty = \frac{1}{T}\int_T |x(t)|^2 \, dt = \frac{1}{2}\left[\int_0^1 K^2 \, dt + \int_1^2 (-K)^2 \, dt\right]$$
$$= K^2 \quad ■$$

**Comment** The signal energy as defined in (2.7) or (2.8) does not indicate the actual energy of the signal because the signal energy depends not only on the signal but also on the load. To make this point clearer assume we have a voltage signal $v(t)$ across a resistor $R$, the actual energy delivered to the resistor by the voltage signal would be

$$Energy = \int_{-\infty}^{\infty} \frac{|v(t)|^2}{R} \, dt = \frac{1}{R}\int_{-\infty}^{\infty} |v(t)|^2 \, dt = \frac{E_\infty}{R}$$

The signal energy is *proportional* to the actual physical energy delivered by the signal and the proportionality constant, in this case, is $R$. However, one can always interpret signal energy as the energy dissipated in a normalized load of a $1\Omega$ resistor. Furthermore, the units of the signal energy depend on the units of the signal. For the voltage signal whose unit is volt(V), the signal energy units is expressed in $V^2.s$ (Voltage squared-seconds). Parallel observations applies to signal power defined in (2.11). ■

Signals which have finite signal energy are referred to as *energy signals* and signals which have infinite signal energy but finite average signal power are referred to as *power signals*. Observe that power is the time average of energy. Since the averaging is over an infinitely large interval, a signal with finite energy has zero power, and a signal with finite power has infinite energy. Therefore, a signal cannot both be an energy and power signal. On the other hand, there are signals that are neither energy nor power signals. The ramp signal (see section 2.3.4) is such an example. Figure 2.6 Shows examples of CT and DT energy and power signals.

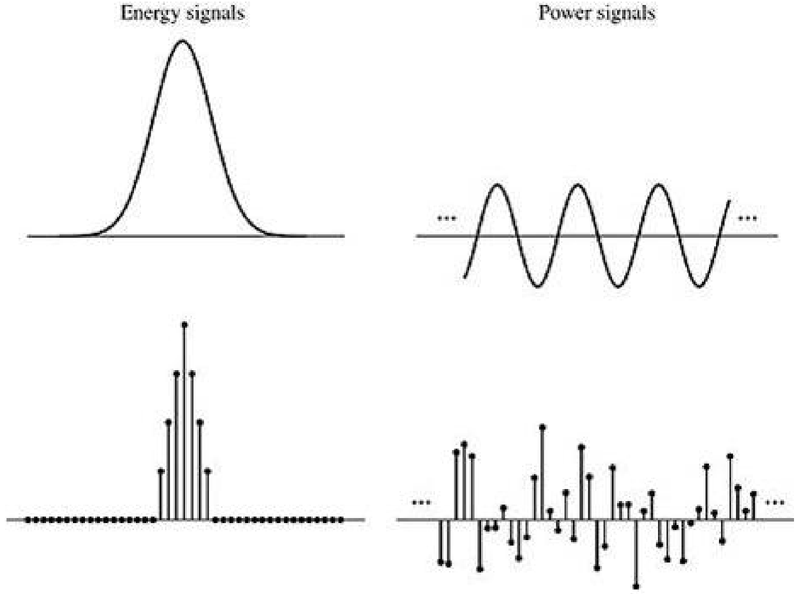



**Figure 2.6:** Examples of CT and DT energy and power signals.

### 2.1.4 Even and Odd Functions

A function $x(t)$ is said to be an *even function* of $t$ if

$$x(t) = x(-t) \quad \text{for all } t \tag{2.17}$$

and a function $x(t)$ is said to be an *odd function* of $t$ if

$$x(t) = -x(-t) \quad \text{for all } t \tag{2.18}$$

An even function has the same value at the instants $t$ and -$t$ for all values of $t$. Clearly, $x(t)$ in this case is symmetrical about the vertical axis (the vertical axis acts as a mirror) as shown in Figure 2.7. On the other hand, the value of an odd function at the instant $t$ is the negative of its value at the instant -$t$. Therefore, $x(t)$ in this case is anti-symmetrical about the vertical axis, as depicted in Figure 2.7. The most important even and odd functions in signal analysis are cosines and sines. Cosines are even, and sines are odd.

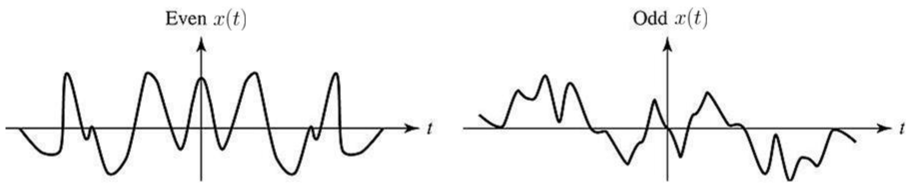



**Figure 2.7:** An even and odd function of $t$.

**Example 2.6** Show that every function $x(t)$ can be decomposed into two components, an even component $x_e(t)$ and an odd component $x_o(t)$.

■ **Solution** Let the signal $x(t)$ be expressed as a sum of its two components $x_e(t)$ and $x_o(t)$ as follows

$$x(t) = x_e(t) + x_o(t)$$

Define $x_e(t)$ to be even and $x_o(t)$ to be odd; that is $x_e(t) = x_e(-t)$ from (2.17) and $x_o(t) = -x_o(-t)$ from (2.18). Putting $t = -t$ in the expression for $x(t)$, we may then write

$$\begin{aligned} x(-t) &= x_e(-t) + x_o(-t) \\ &= x_e(t) - x_o(t) \end{aligned}$$

Solving for $x_e(t)$ and $x_o(t)$, we thus obtain

$$x_e(t) = \tfrac{1}{2}[x(t) + x(-t)] \tag{2.19}$$

and

$$x_o(t) = \tfrac{1}{2}[x(t) - x(-t)] \quad ■ \tag{2.20}$$

The above definitions of even and odd signals assume that the signals are real valued. In the case of a complex-valued signal, we may speak of conjugate symmetry. A complex-valued signal $x(t)$ is said to be *conjugate symmetric* if it satisfies the condition $x(-t) = x^*(t)$, where $x^*(t)$ denotes the complex conjugate of $x(t)$.

### Some properties of Even and Odd Functions

Even and odd functions have the following properties:

- For even functions, $x(t)$ is symmetrical about the vertical axis, it follows from Figure 2.8 that $\int_{-a}^{a} x(t)\,dt = 2\int_{0}^{a} x(t)\,dt$.
- For odd functions, it is also clear from Figure 2.8 that $\int_{-a}^{a} x(t)\,dt = 0$.

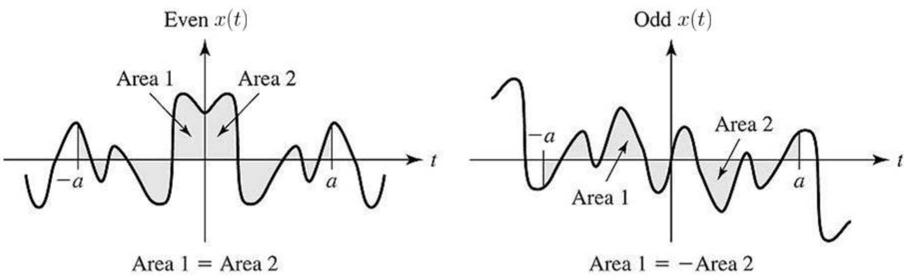
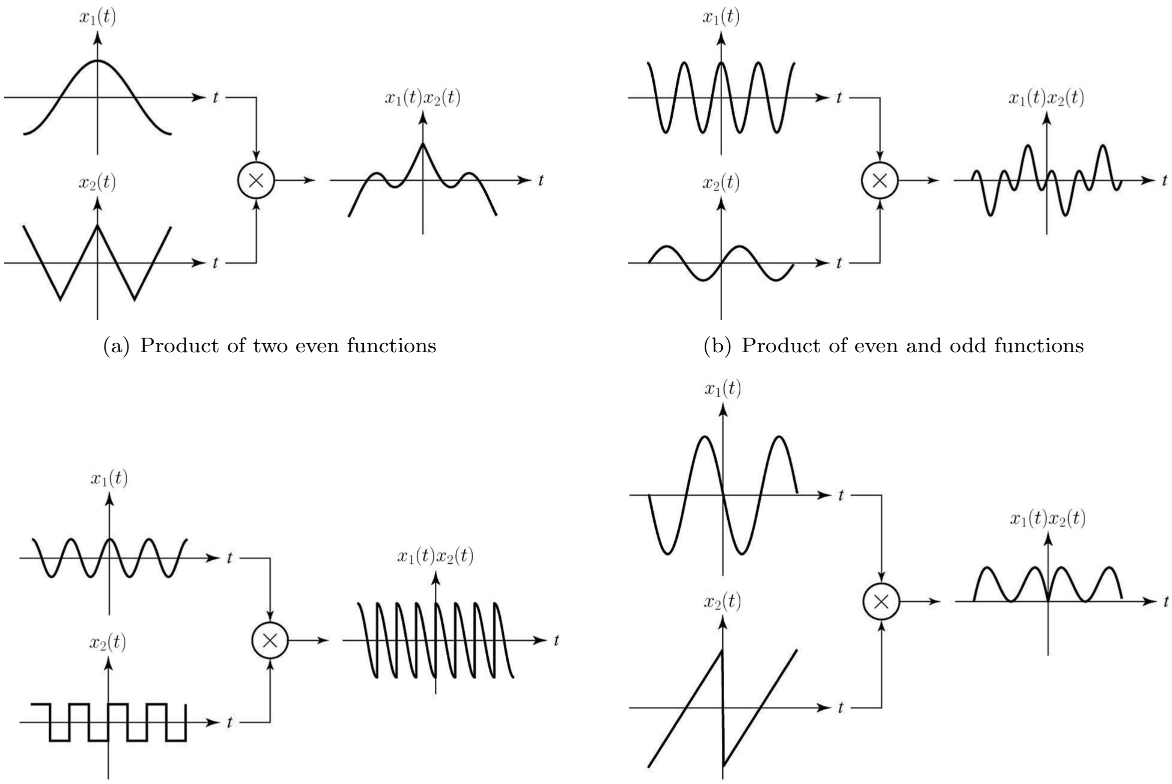



**Figure 2.8:** Integrals of even and an odd function.

- Even function × even function = even function, Figure 2.9(a).
- Even function × odd function = odd function, Figure 2.9(b) and (c).
- Odd function × odd function = even function, Figure 2.9(d).




(a) Product of two even functions
(b) Product of even and odd functions
(c) Product of even and odd functions
(d) Product of two odd functions

**Figure 2.9:** Examples of products of even and odd CT functions.

- If the odd part of a function is zero, the function is even.
- If the even part of a function is zero, the function is odd.

The proofs of these facts are trivial and follow directly from the definition of odd and even functions in (2.17) and (2.18).

**Example 2.7** Find the even and odd components of $x(t) = e^{jt}$.

■ **Solution** This function can be expressed as a sum of the even and odd components $x_e(t)$ and $x_o(t)$, we obtain

$$e^{jt} = x_e(t) + x_o(t)$$

where form (2.19) and (2.20)

$$x_e(t) = \frac{1}{2}[e^{jt} + e^{-jt}] = \cos t$$

and

$$x_o(t) = \frac{1}{2}[e^{jt} - e^{-jt}] = j \sin t \quad \blacksquare$$

## 2.2 Useful Signal Operations

In signal and system analysis it is important to be able to describe signals both analytically and graphically and to be able to relate the two different kinds of descritptions to each other. We begin by considering three useful signal operations or transformations in time: *shifting* (also called *time translation*), *scaling* and *reflection.* All theses operations involve transformations of the independent variable. Later in this section we consider transformations performed on the dependent variable, namely, *amplitude trransformations.*

### 2.2.1 Time Shifting

Consider a signal $x(t)$ as shown in Figure 2.10(a) and the same signal delayed by $T$ seconds as illustrated in Figure 2.10(b) which we shall denote $\phi(t)$. Whatever happens in $x(t)$ at some instant $t$ also happens in $\phi(t)$ but $T$ seconds later at the instant $t+T$. Therefore

$$\phi(t+T) = x(t)$$

and

$$\phi(t) = f(t-T).$$

Therefore, to time shift a signal by $T$, we replace $t$ with $t-T$. Thus $x(t-T)$ represents $x(t)$ time shifted by $T$ seconds. If $T$ is positive, the shift is to the right (delay). If $T$ is negative, the shift is to the left (advance). Time shifting occur in many real physical systems, such as radar and communication systems.

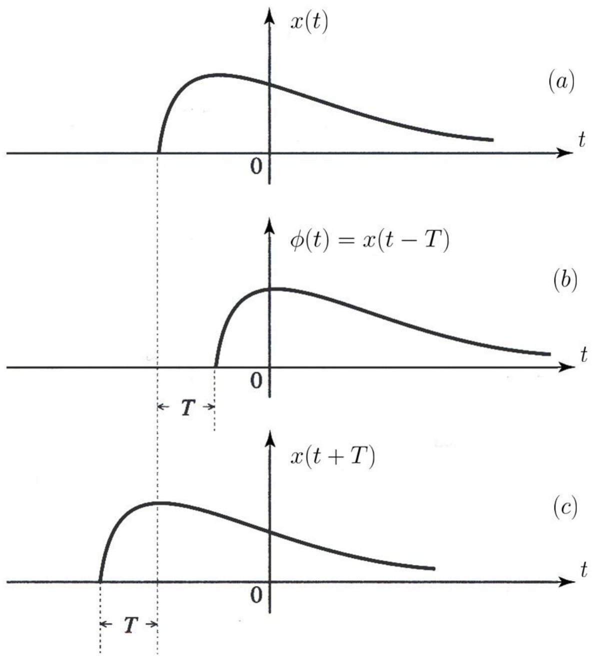



**Figure 2.10:** Time shifting a signal

**Example 2.8** Let the graph in Figure 2.11 defines a signal $x(t)$, sketch $x(t-1)$.

**Table 2.1:** Selected values of $x(t-1)$.

| $t$ | $t-1$ | $x(t-1)$ |
|---|---|---|
| -4 | -5 | 0 |
| -3 | -4 | 0 |
| -2 | -3 | -3 |
| -1 | -2 | -5 |
| 0 | -1 | -4 |
| 1 | 0 | -2 |
| 2 | 1 | 0 |
| 3 | 2 | 4 |
| 4 | 3 | 1 |

■ **Solution** We can begin to understand how to make this transformation by computing the values of $x(t-1)$ at a few selected points as shown in Table 2.1. It should be apparent that replacing $t$ by $t-1$ has the effect of shifting the function one unit to the right as in Figure 2.12. ■

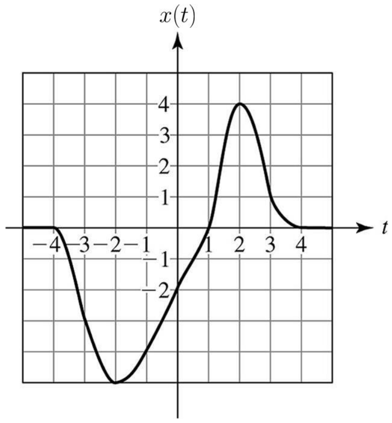



**Figure 2.11:** Graphical definition of a CT function $x(t)$.

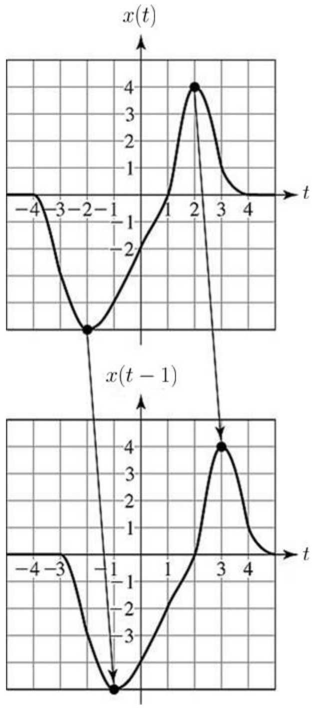



**Figure 2.12:** Graph of $x(t-1)$ in relation to $x(t)$.

## 2.2.2 Time Scaling

The compression or expansion of a signal in time is known as *time scaling.* Consider the signal $x(t)$ of Figure 3.2, if $x(t)$ is to be expanded (stretched) in time by a factor $a(a < 1)$, the resulting signal $\phi(t)$ is given by

$$\phi(t) = x(at)$$

**Table 2.2:** Selected values of $x(\frac{t}{2})$.

| $t$ | $\frac{t}{2}$ | $x(\frac{t}{2})$ |
|---|---|---|
| -4 | -2 | -5 |
| -2 | -1 | -4 |
| 0 | 0 | -2 |
| 2 | 1 | 0 |
| 4 | 2 | 4 |

Assume $a = \frac{1}{2}$, then $\phi(t) = x(\frac{t}{2})$, constructing a similar table to Table 2.1, paying particular attention to the turning points of the original signal, as shown in Table 2.2. Next, plot $x(\frac{t}{2})$ as function of $t$ as illustrated in Figure 2.13. On the other hand, if $x(t)$ is to be compressed in time then $a > 1$. In summary, to time scale a signal by a factor $a$, we replace $t$ with $at$. If $a > 1$, the scaling results in compression, and if $a < 1$, the scaling results in expansion. If we think of $x(t)$ as an output when listening to an answering machine, then $x(3t)$ is the signal obtained when listening to the messages on fast forward at three times the speed at which it was recorded. This speeds up the signal in time and increases the frequency content of the speaker's voice. Similarly, $x(\frac{t}{2})$ is the signal obtained when the answering machine is played back at half speed.

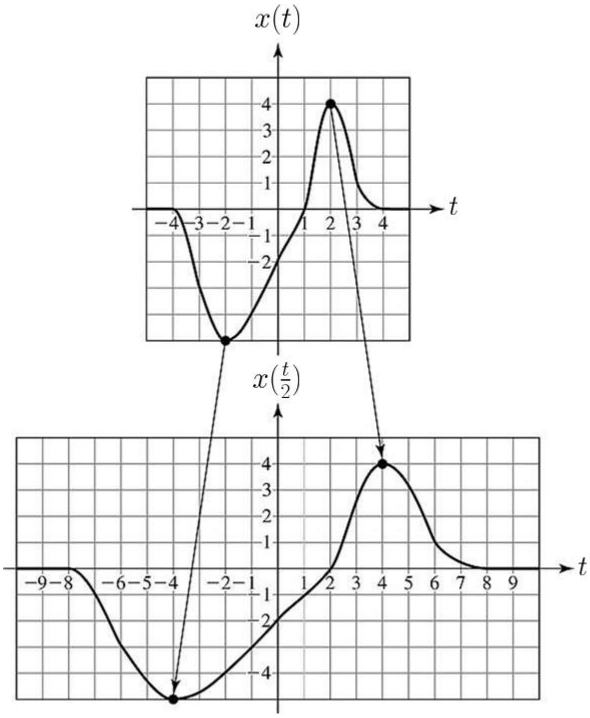



**Figure 2.13:** Graph of $x(\frac{t}{2})$ in relation to $x(t)$.

### 2.2.3 Time Reflection

Also called *time reversal*, in order to obtain a reflected version of a signal $x(t)$ we simply replace $t$ with $-t$ in the original signal $x(t)$. Hence,

$$\phi(t) = x(-t)$$

where $\phi(t)$ denotes the transformed signal. Observe that whatever happens at the time instant $t$ also happens at the instant $-t$. $\phi(t)$ is the mirror image of $x(t)$ about the vertical axis. Thus if $x(t)$ represents the output signal when listening to the message on an answering machine, then $x(-t)$ is the signal when listening to the message when the rewind switch is pushed on (assuming that the rewind and play speeds are the same).

### 2.2.4 Amplitude Transformations

We now consider signal amplitude transformations, unlike time transformations, they are transformations of the dependent variable. Amplitude transformations follow the same rules as time transformations. The three transformations in amplitude are of the general form

$$\phi(t) = Ax(t) + C$$

where $A$ and $C$ are constants. For example, consider $\phi(t) = -2x(t) - 1$. The value $A = -2$ yields amplitude reflection (the minus sign) and amplitude scaling ($|A| = 2$), and $C = -1$ shifts the amplitude of the signal. Amplitude scaling and amplitude reflection occur in many real physical systems. An amplifier for example invert the input signal in addition to amplifying the signal. Some amplifiers not only amplify signals, but also add (or remove) a constant, or dc, value.

### 2.2.5 Multiple Transformations

All time and amplitude transformation of a signal can be applied simultaneously, for example

$$\phi(t) = Ax\left(\frac{t - t_0}{a}\right) + C \tag{2.21}$$

To understand the overall effect, it is usually best to break down a transformation like (2.21) into successive simple transformations, (without any loss of generality we will assume $C = 0$)

$$x(t) \xrightarrow[\text{scaling A}]{\text{Amplitude}} Ax(t) \xrightarrow{t \to t/a} Ax\left(\frac{t}{a}\right) \xrightarrow{t \to t - t_0} Ax\left(\frac{t - t_0}{a}\right) \tag{2.22}$$

Observe here that the order of the transformation is important. For example, if we exchange the order of the time-scaling and time-shifting operations in (2.22), we get

$$x(t) \xrightarrow[\text{scaling A}]{\text{Amplitude}} Ax(t) \xrightarrow{t \to t - t_0} Ax(t - t_0) \xrightarrow{t \to t/a} Ax\left(\frac{t}{a} - t_0\right) \neq Ax\left(\frac{t - t_0}{a}\right)$$

The result of this sequence of transformations is different from the preceding result. We could have obtained the same preceding result if we first observe that

$$Ax\left(\frac{t - t_0}{a}\right) = Ax\left(\frac{t}{a} - \frac{t_0}{a}\right)$$

Then we could time-shift first and time-scale second, yielding

$$x(t) \xrightarrow[\text{scaling A}]{\text{Amplitude}} Ax(t) \xrightarrow{t \to t - \frac{t_0}{a}} Ax\left(t - \frac{t_0}{a}\right) \xrightarrow{t \to t/a} Ax\left(\frac{t}{a} - \frac{t_0}{a}\right) = Ax\left(\frac{t - t_0}{a}\right)$$

For a different transformation, a different sequence may be better, for example

$$Ax(bt - t_0)$$

In this case the sequence of time shifting and then time scaling is the simplest path to correct transformation

$$x(t) \xrightarrow[\text{scaling A}]{\text{Amplitude}} Ax(t) \xrightarrow{t \to t - t_0} Ax(t - t_0) \xrightarrow{t \to bt} Ax(bt - t_0)$$

In summary, we defined six transformations (a) shifting, scaling, and reflection with respect to time; and (b) reflection, scaling, and shifting with respect to amplitude. All six transformations have applications in the field of signal and system analysis.

**Example 2.9** Let a signal be defined graphically as shown in Figure 2.14. Find and sketch

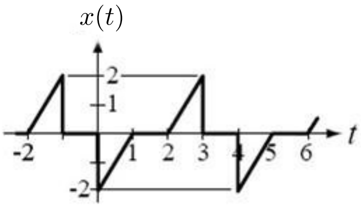



**Figure 2.14:** Plot of $x(t)$ for Example 2.9.

the transformed function

$$\phi(t) = -2x\left(\frac{t-1}{2}\right)$$

**Table 2.4:** Selected values of $-2x(\frac{t-1}{2})$.

| $t$ | $t' = \frac{t-1}{2}$ | $-2x(\frac{t-1}{2})$ |
|---|---|---|
| -1 | -1 | -4 |
| 1 | 0 | 4 |
| 3 | 1 | 0 |
| 5 | 2 | 0 |
| 7 | 3 | -4 |

■ **Solution** Construct a table to compute the values of $-2x(\frac{t-1}{2})$ at a few selected points as shown in Table 2.4. Note that $x(t)$ is periodic with fundamental period 4, therefore, $x(t) = x(t+4)$. Next, plot $-2x(\frac{t-1}{2})$ as function of $t$. ■

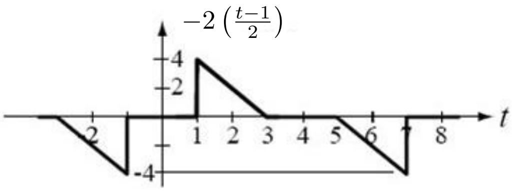



**Figure 2.15:** Plot of $x(t)$ for Example 2.9.

**Remark:** When solving using the method of constructing a table as the one shown in Table 2.4, it is much easier to start constructing your table from the second column i.e. the time transformation argument of the function. The time transformation argument in this example is $\frac{t-1}{2}$ which can be labeled $t'$. Start with few selected points of $t'$, find the corresponding $t$ points and fill the column corresponding to $t$. This could be done easily by writing an expression of $t$ in terms of $t'$, $t = 2t' + 1$. Finally, plot $-2x(\frac{t-1}{2})$ as function of $t$. ■

The same result could have been obtained by doing the transformation graphically paying particular attention to the correct sequence of transformations. We can consider the following sequence

$$x(t) \xrightarrow[A=-2]{\substack{\text{Amplitude}\\\text{transformation}}} -2x(t) \xrightarrow{t \to t/2} -2x\left(\frac{t}{2}\right) \xrightarrow{t \to t-1} -2x\left(\frac{t-1}{2}\right)$$

Alternatively,

$$x(t) \xrightarrow[A=-2]{\substack{\text{Amplitude}\\\text{transformation}}} -2x(t) \xrightarrow{t \to t-\frac{1}{2}} -2x\left(t-\frac{1}{2}\right) \xrightarrow{t \to t/2} -2x\left(\frac{t}{2}-\frac{1}{2}\right)$$

which also leads to the same result.

## 2.3 Useful Signal Functions

### 2.3.1 Complex Exponentials and Sinusoids[3]

Some of the most commonly used mathematical functions for describing signals should already be familiar: the CT sinusoids

$$x(t) = A\,cos\left(\frac{2\pi t}{T_0} + \phi\right) = A\,cos(\omega_0 t + \phi) = A\,cos(2\pi f_0 t + \phi)$$

where

- A = amplitude of sinusoid or exponential
- $T_0$ = real fundamental period of sinusoid
- $f_0$ = real fundamental frequency of sinusoid, Hz
- $\omega_0$ = real fundamental frequency of sinusoid, radians per second (rad/s)

Another important function in the area of signals and systems is the exponential function

$$x(t) = Ce^{at}$$

where both $C$, and $a$ can be real or complex. In particular we are interested in signals of the complex exponential form $x(t) = e^{j\omega_o t}$. An important relation that is often applied in analyses which involve complex exponential functions is *Euler's relation*, given by

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{2.23}$$

Replacing $\theta$ in (2.23) with $-\theta$ yields

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta \tag{2.24}$$

since the cosine function is even and the sine function is odd. The sum of (2.23) and (2.24) can be expressed as

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{2.25}$$

and the difference of (2.23) and (2.24) can be expressed as

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{2.26}$$

The complex exponential in (2.23) can also be expressed in polar form as

$$e^{j\theta} = 1\angle\theta \tag{2.27}$$

where the notation $R\angle\theta$ signifies the complex function of magnitude $R$ at the angle $\theta$. Three cases for exponential functions will now be investigated.

#### CASE 1 ($C$ and $a$ Real)

Here, both $C$ and $a$ are real for the exponential $x(t) = Ce^{at}$. The signal $x(t) = Ce^{at}$ is plotted in Figure 2.16 for $C > 0$ with $a > 0$, $a < 0$, and $a = 0$. For $a > 0$, the signal is growing without limit with increasing time. For $a < 0$, the signal is decaying toward zero as time increases. For $a = 0$, the signal is constant.

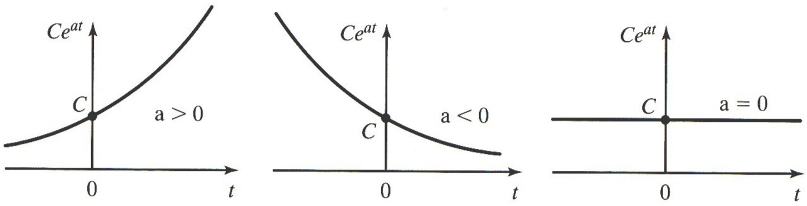



**Figure 2.16:** Exponential signals.

CASE 2 ($C$ Complex, $a$ Imaginary)

Next we consider the case that $C$ is complex and $a$ is imaginary, namely,

$$x(t) = Ce^{at}; \quad C = Ae^{j\phi}, \quad a = j\omega_0$$

where $A$, $\phi$, and $\omega_o$ are real constants. The complex exponential signal $x(t)$ can be expressed as

$$\begin{aligned} x(t) &= Ae^{j\phi}e^{j\omega_o t} = Ae^{j(\omega_o t+\phi)} \\ &= A\cos(\omega_o t + \phi) + jA\sin(\omega_o t + \phi) \end{aligned}$$

from Euler's relation in (2.23). When the argument of the exponential function is purely imaginary, as in this case, the resulting complex exponential is called a *complex sinusoid* because it contains a cosine and a sine as its real and imaginary parts as illustrated in Figure 2.17 for a complex sinusoid in time ($\phi$ is assumed zero for simplicity). The projection of the complex sinusoid onto a plane parallel to the plane containing the real and $t$ axes is the cosine function, and the projection onto a plane parallel to the plane containing the imaginary and $t$ axes is the sine function.

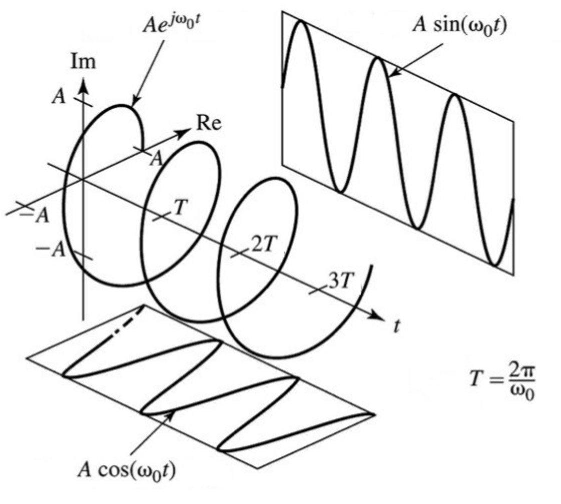



**Figure 2.17:** Relation between a complex sinusoid and a real sine and a real cosine.

Physically, the function $x(t) = e^{j\omega_o t}$ may be thought of as describing the motion of a point on the rim of a wheel of unit radius. The wheel revolves counter-clockwise at an angular rate of $\omega_o$ radians per second. From trigonometry (see Figure 2.18), we see that the projection of the point on the real axis is $\cos \omega_o t$, and the projection on the imaginary axis is $\sin \omega_o t$.

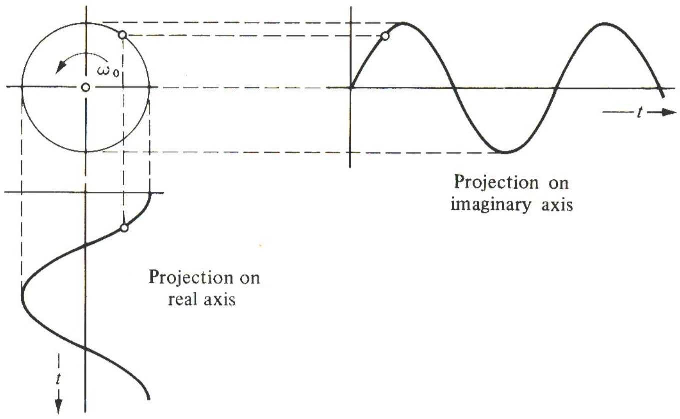



**Figure 2.18:** Complex notation for angular signals.

Combining, we can write

$$e^{j\omega_o t} = \cos \omega_o t + j \sin \omega_o t$$

to obtain the same plot as in Figure 2.17. Note, since we assumed a wheel of unit radius, implies cosine and sine projections with unit amplitude, i.e., $A = 1$.

CASE 3 ( Both $C$ and $a$ Complex)

For this case, the complex exponential $x(t) = Ce^{at}$ has the parameters

$$x(t) = Ce^{at}; \quad C = Ae^{j\phi}; \quad a = \sigma + j\omega_o$$

where $A$, $\phi$, $\sigma$, and $\omega_o$ are real constants. The complex exponential signal can then be expressed as

$$\begin{aligned} x(t) &= Ae^{j\phi}e^{(\sigma+j\omega_o)t} = Ae^{\sigma t}e^{j(\omega_o t+\phi)} \\ &= Ae^{\sigma t}\cos(\omega_o t + \phi) + jAe^{\sigma t}\sin(\omega_o t + \phi) \end{aligned}$$

Plots of the real part of $x(t)$, i.e, $Re[x(t)] = Ae^{\sigma t}\cos \omega_o t$ are given in Figure 2.19 for $\phi = 0$. Figure 2.19(a) shows the case that $\sigma > 0$. Figure 2.19(b) shows the case that $\sigma < 0$; this signal is called a *damped sinusoid.* In Figure 2.19(a), by definition, the *envelope* of the signal is $\pm Ae^{\sigma t}$, and $\sigma$ is sometimes called the *exponential damping coefficient.* For $\sigma = 0$ the signal is called an *undamped sinusoid.*

The signals defined in this section appear in the responses of a wide class of physical systems. In terms of circuit analysis, the real exponential of Case 1 appears in the transient, or natural, response of $RL$ and $RC$ circuits. The undamped response of Case 2 appears in the transient response of $LC$ circuits, and the damped sinusoid of case 3 can appear in the transient response of $RLC$ circuits.

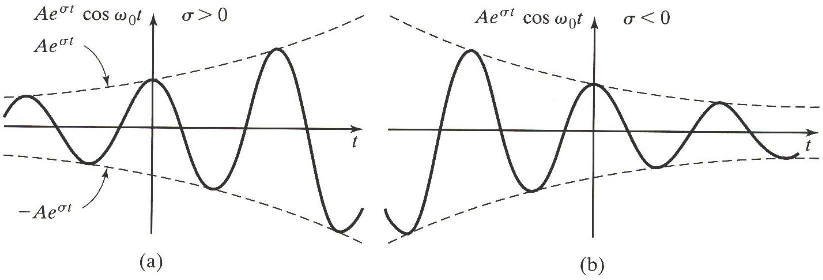



**Figure 2.19:** Real part of complex exponential.

### Some properties of Complex Exponential Functions

The complex exponential function $x(t) = e^{j(\omega_0 t+\phi)}$ has a number of important properties:

1. It is periodic with fundamental period $T = \frac{2\pi}{|\omega_0|}$.

   **Proof** We show that $x(t) = x(t+kT)$, therefore,

   $$\begin{aligned} e^{j(\omega_0 t+\phi)} &= e^{j\left(\omega_0\left(t+\frac{2\pi k}{\omega_0}\right)+\phi\right)} \\ &= e^{j(\omega_o t+\phi)} \quad \text{for any} \quad k \in \mathcal{Z} \end{aligned}$$

   since $e^{j2\pi k} = 1$ for any $k \in \mathcal{Z}$. ■

2. $\text{Re}\{e^{j(\omega_0 t+\phi)}\} = \cos(\omega_0 t+\phi)$ and the $\text{Im}\{e^{j(\omega_0 t+\phi)}\} = \sin(\omega_0 t+\phi)$, these terms are real sinusoids of frequency $\omega_0$.

3. The term $\phi$ is often called the *phase*. Note that we can write

   $$e^{j(\omega_0 t+\phi)} = e^{j\omega_0\left(t+\frac{\phi}{\omega_0}\right)}$$

   which implies that the phase has the effect of time shifting the signal.

4. Since complex exponential functions are periodic they have infinite total energy but finite power, thus,

   $$P_\infty = \frac{1}{T}\int_T \left|e^{j\omega_0 t}\right|^2 dt = \frac{1}{T}\int_t^{t+T} 1\, d\tau = 1$$

5. Set of periodic exponentials with fundamental frequencies that are multiples of a single positive frequency $\omega_0$ are said to be *harmonically* related complex exponentials

   $$\Phi_k(t) = e^{jk\omega_0 t} \quad for\ k = 0, \pm 1, \pm 2, \cdots \tag{2.28}$$

   - If $k = 0 \Rightarrow \Phi_k(t)$ is a constant.
   - If $k \neq 0 \Rightarrow \Phi_k(t)$ is periodic with fundamental frequency $|k|\,\omega_0$ and fundamental period $\frac{2\pi}{|k|\omega_0} = \frac{T}{|k|}$. Note that each exponential in (2.28) is also periodic with period $T$.

- $\Phi_k(t)$ is called the $k^{th}$ *harmonic.* Harmonic (from music): tones resulting from variations in acoustic pressures that are integer multiples of a fundamental frequency.

We will make extensive use of harmonically related complex exponentials later when we study the Fourier series representation of periodic signals.

### 2.3.2 The Unit Step Function

A CT *unit step* function is defined as, (Figure 2.20)

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{2.29}$$

This function is called the unit step because the height of the step change in function value is one unit in the system of units used to describe the signal. The

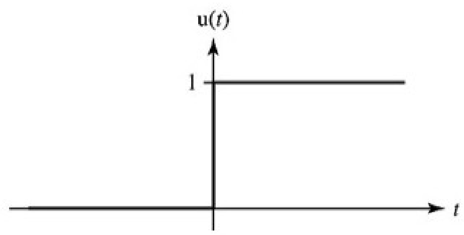



**Figure 2.20:** The CT unit step function

function is discontinuous at $t = 0$ since the function changes instantaneously from 0 to 1 when $t = 0$. Some authors define the unit step by

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \qquad or \qquad u(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

For most analysis purposes these definitions are all equivalent. The unit step is defined and used in signal and system analysis because it can mathematically represent a very common action in real physical systems, fast switching from one state to another. For example in the circuit shown in Figure 2.21 the switch

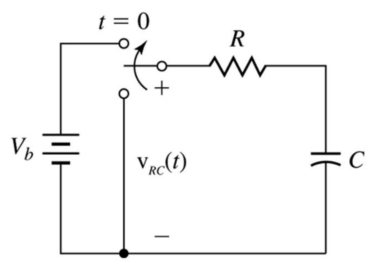



**Figure 2.21:** Circuit with a switch whose action can be represented mathematically by a unit step

moves from one position to the other at time $t = 0$. The voltage applied to the $RC$ circuit can be described mathematically by $v_0(t) = v_s(t)u(t)$.

The unit step function is very useful in specifying a function with different mathematical description over different intervals.

**Example 2.10**

Consider, the rectangular pulse depicted in Figure 2.22, express such a pulse in terms of the unit step function.

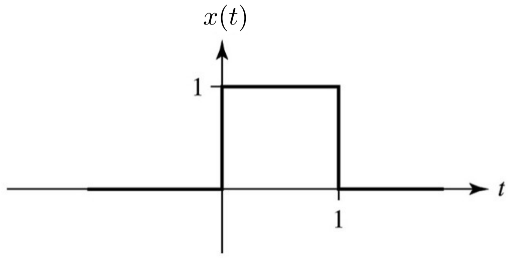



**Figure 2.22:** A rectangular pulse.

■ **Solution** Observe that the pulse can be expressed as the sum of two delayed unit step functions. The unit step function delayed by $t_0$ seconds is $u(t - t_0)$. Therefore,

$$g(t) = u(t) - u(t-1) \quad ■$$

The DT Unit Step Function

The DT counterpart of the CT unit step function $u(t)$ is $u[n]$, also called unit sequence (Figure 2.23) defined by

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{2.30}$$

For this function there is no disagreement or ambiguity about its value at $n = 0$, it is one.

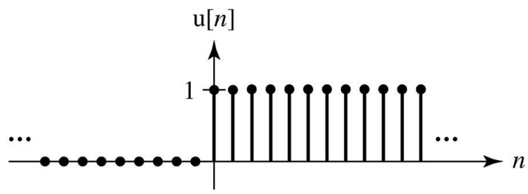



**Figure 2.23:** The DT unit step function.

### 2.3.3 The Signum Function

The signum function illustrated in Figure 2.24 is closely related to the unit step function. It is some time called the *sign* function, but the name *signum* is more common so as not confuse the sounds of the two words *sign* and *sine* !! The signum function is defined as

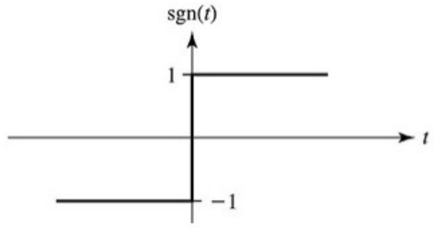



**Figure 2.24:** The CT signum function.

$$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases} \tag{2.31}$$

and can be expressed in terms of the unit step function as

$$\mathrm{sgn}(t) = 2u(t) - 1$$

### 2.3.4 The Unit Ramp Function

Another type of signal that occurs in systems is one which is switched on at some time and changes linearly after that time or one which changes linearly before some time and is switched off at that time. Figure 2.25 illustrates some examples. Signals of this kind can be described with the use of the *ramp* function. The CT unit ramp function (Figure 2.26) is the integral of the unit step

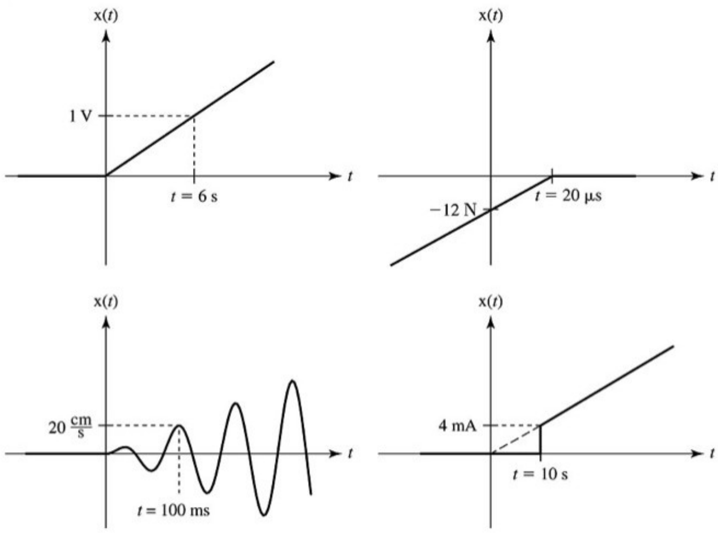



**Figure 2.25:** Functions that change linearly before or after some time or that are multiplied by functions that change linearly before or after some time.

function. It is called the unit ramp function because, for positive $t$, its slope is

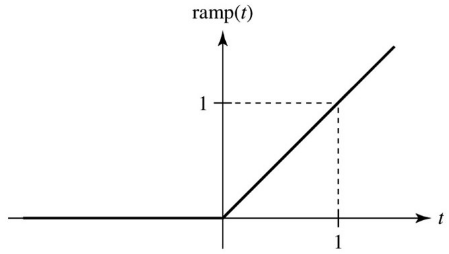



**Figure 2.26:** The CT unit ramp function.

one.

$$\text{ramp}(t) = \begin{cases} t, & t > 0 \\ 0, & t \leq 0 \end{cases} = \int_{-\infty}^{t} u(\lambda)\, d\lambda = tu(t) \quad (2.32)$$

The integral relationship in (2.32) between the CT unit step and CT ramp functions is shown below in Figure 2.27.

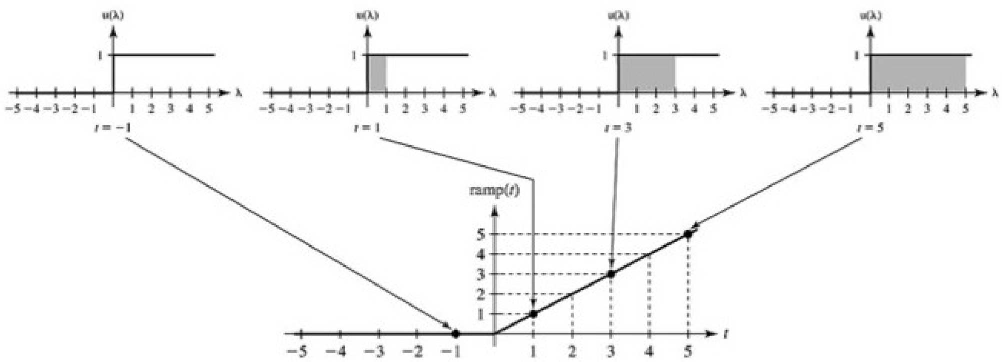



**Figure 2.27:** Illustration of the integral relationship between the CT unit step and the CT unit ramp.

## 2.3.5 The Rectangle Function

A very common type of signal occurring in systems is one in which a signal is switched on at some time and then back off at a later time. The rectangle function (Figure 2.28) is defined as

$$\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1, & |t| < \frac{\tau}{2} \\ 0, & otherwise \end{cases} \quad (2.33)$$

Use of this function shortens the notation when describing some complicated signals. The notation used in (2.33) is convenient, $\tau$ represent the width of the rectangle function while the rectangle centre is at zero, therefore any time transformations can be easily applied to the notation in (2.33). A special case of the rectangle function defined in (2.33) is when $\tau = 1$, it is called the *unit*

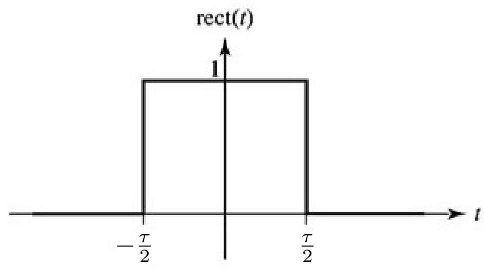



**Figure 2.28:** The CT rectangle function.

*rectangle* function, rect($t$), (also called the *square pulse*). It is a unit rectangle function because its width, height, and area are all one.

**Example 2.11** Write down a mathematical expression to describe the time shifted rectangular pulse $x(t)$ shown in Figure 2.29.

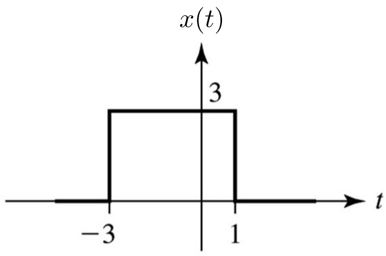



**Figure 2.29:** A time shifted rectangular pulse.

■ **Solution** It can be expressed as three different functions of unit step signals:

$$x(t) = \begin{cases} 3u(t+3) - 3u(t-1) \\ 3u(1-t) - 3u(-3-t) \\ 3u(1-t)u(t+3) \end{cases}$$

However, it is much more convenient to simply describe it as

$$x(t) = 3 \text{ rect}\left(\frac{t+1}{4}\right) \quad \blacksquare$$

The rectangular pulse is useful in extracting part of a signal. For example, the signal $x(t) = \cos t$ has a period $T = 2\pi$. Consider a signal composed of one period of this cosine function begining at $t = 0$, and zero for all other time. This signal can be expressed as

$$x(t) = (\cos t)[u(t) - u(t - 2\pi)] = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

The rectangular pulse notation allows us to write

$$x(t) = \cos t \text{ rect}\left(\frac{t-\pi}{2\pi}\right)$$

This sinusoidal pulse is plotted in Figure 2.30.

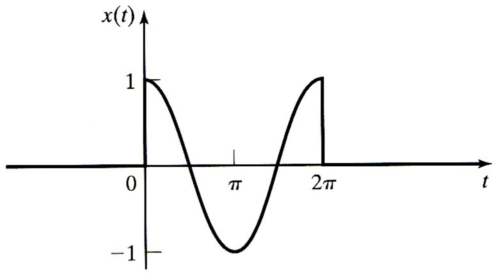



**Figure 2.30:** The function $x(t) = \cos t \, \mathrm{rect}\left(\frac{t-\pi}{2\pi}\right)$

**Example 2.12**

Express the signal shown in Figure 2.31 in terms of unit step functions.

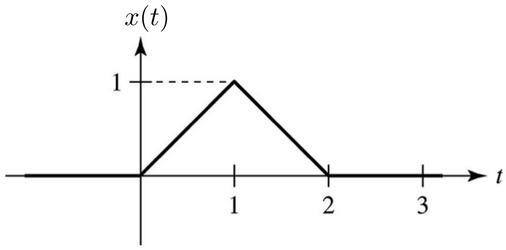



**Figure 2.31:** Signal for Example 2.12.

■ **Solution** The signal $x(t)$ can be conveniently handled by writing an equation for each segment of the signal as follows

$$x(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \end{cases}$$

The signal in the interval $0 < t < 1$ can be written as $t[u(t) - u(t-1)]$. Similarly, the part between $1 < t < 2$ can be represented as $(2 - t)[u(t-1) - u(t-2)]$. therefore, one possible respresentation for $x(t)$ is

$$x(t) = tu(t) - 2(t-1)[u(t-1)] + (t-2)[u(t-2)] \quad ■$$

### 2.3.6 The Unit Impulse Function

The unit impulse function $\delta(t)$, also called the *delta function* is one of the most important functions in the study of signals and systems and yet the strangest. It was first defined by P.A.M Dirac (sometimes called by his name the *Dirac Distribution*) as

$$\delta(t) = 0 \qquad t \neq 0 \tag{2.34}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{2.35}$$

Try to visualise this function: a signal of unit area equals zero everywhere except at $t = 0$ where it is undefined !! To be able to understand the definition of the delta function let us consider a unit-area rectangular pulse defined by the function

$$\delta_a(t) = \begin{cases} \frac{1}{a}, & |t| < \frac{a}{2} \\ 0, & |t| > \frac{a}{2} \end{cases} \tag{2.36}$$

and is illustrated in Figure 2.32. Now imagine taking the limit of the function

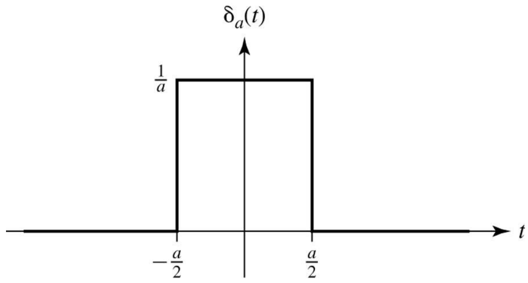



**Figure 2.32:** A unit-area rectangular pulse of width $a$.

$\delta_a(t)$ as $a$ approaches zero. Try to visualise what will happen, the width of the rectangular pulse will become infinitesimally small, a height that has become infinitely large, and an overall area that has been maintained at unity. Using this approach to approximate the unit impulse which is now defined as

$$\delta(t) = \lim_{a \to 0} \delta_a(t) \tag{2.37}$$

Other pulses, such as triangular pulse may also be used in impulse approximations (Figure 2.33). The area under an impulse is called its *strength*, or sometimes its *weight*. An impulse with a strength of one is called a unit impulse. The impulse cannot be graphed in the same way as other functions because its amplitude is undefined when $t = 0$. For this reason a unit impulse is represented by a vertical arrow a spear-like symbol. Sometimes, the strength of the impulse is written beside it in parentheses, and sometimes the height of the arrow indicates the strength of the impulse. Figure 2.34 illustrates some ways of representing impulses graphically.

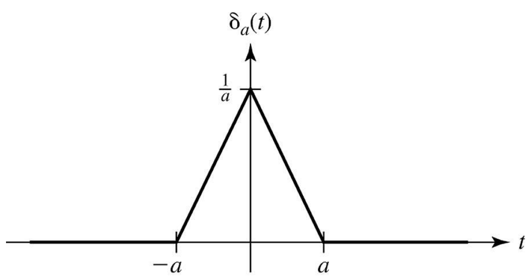



**Figure 2.33:** A unit area triangular pulse.

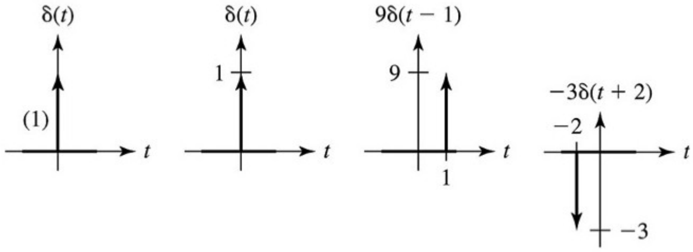



**Figure 2.34:** Graphical representation of impulses.

## 2.3.7 Some Properties of the Unit Impulse

### The Sampling Property

A common mathematical operation that occurs in signals and systems analysis is the multiplication of an impulse with another function $x(t)$ that is known to be continuous and finite at $t = 0$. (i.e. $x(t = 0)$ exists and its value is $x(0)$), we obtain

$$x(t)\delta(t) = x(0)\delta(t) \tag{2.38}$$

since the impulse exists only at $t = 0$. Graphically, this property can be illustrated by approximating the impulse signal by the rectangular pulse $\delta_a(t)$ in (2.36). Let this function multiply another function $x(t)$, the result is a pulse whose height at $t = 0$ is $x(0)/a$ and whose width is $a$, as shown in Figure 2.35. In the limit as $a$ approaches zero the pulse becomes an impulse and the strength is $x(0)$. Similarly, if a function $x(t)$ is multiplied by an impulse $\delta(t-t_0)$(impulse located at $t = t_0$), then

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0) \tag{2.39}$$

provided $x(t)$ is finite and continuous at $t = t_0$.

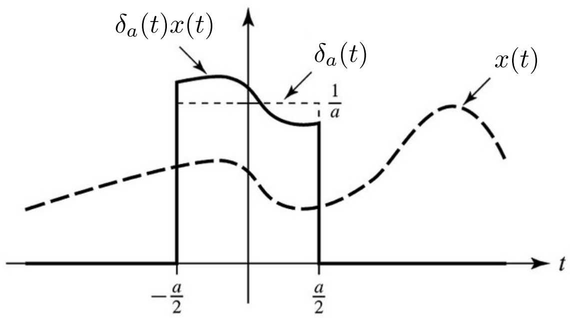



**Figure 2.35:** Multiplication of a unit-area rectangular pulse centered at $t = 0$ and a function $x(t)$, which is continuous and finite at $t = 0$.

### The Sifting Property

*The word sifting is spelled correctly it is not to be confused with the word shifting.*

Another important property that follows naturally from the sampling property is the so-called *sifting* property. In general, the sifting property states that

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_o) \tag{2.40}$$

The result in (2.40) follows naturally from (2.39),

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0)\underbrace{\int_{-\infty}^{\infty} \delta(t-t_0)\,dt}_{=1} = x(t_0) \tag{2.41}$$

This result means that the *area under the product of a function with an impulse $\delta(t-t_o)$ is equal to the value of that function at the instant where the unit impulse is located.* To illustrate this result graphically, consider the unit-area

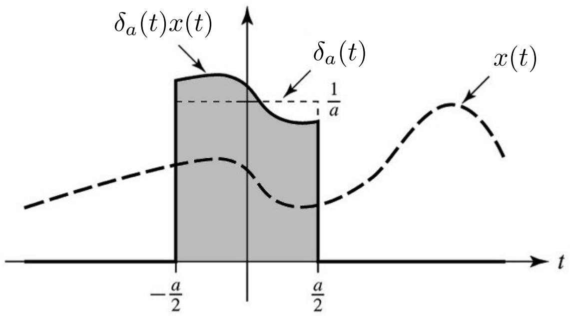



**Figure 2.36:** Graphical illustration of the sifting property.

rectangular function $\delta_a(t)$ defined in (2.36). Let this function multiply another function $x(t)$, which is finite and continuous at $t = 0$, and find the area under the product of the two functions,

$$A = \int_{-\infty}^{\infty} \delta_a(t)x(t)\,dt$$

as illustrated in Figure 2.36. Using the definition of $\delta_a(t)$ we can rewrite the integral as

$$A = \frac{1}{a}\int_{-(a/2)}^{a/2} x(t)\,dt \tag{2.42}$$

Now imagine taking the limit of this integral as $a$ approaches zero. In the limit, the two limits of the integration approach zero from above and below. Since $x(t)$ is finite and continuous at $t = 0$, as $a$ approaches zero in the limit the value of $x(t)$ becomes a constant $x(0)$ and can be taken out of the integral. Then

$$\lim_{a\to 0} A = x(0)\lim_{a\to 0}\frac{1}{a}\int_{-(a/2)}^{a/2} dt = x(0)\lim_{a\to 0}\frac{1}{a}(a) = x(0) \tag{2.43}$$

So in the limit as $a$ approaches zero, the function $\delta_a(t)$ has the interesting property of extracting (hence the name sifting) the value of any continuous finite function $x(t)$ at time $t = 0$ when the multiplication of $\delta_a(t)$ and $x(t)$ is integrated between any two limits which include time $t = 0$. Thus, in other words

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = \lim_{a\to 0}\int_{-\infty}^{\infty} x(t)\delta_a(t)\,dt = x(0) \tag{2.44}$$

The definition of the impulse function (2.34) and (2.35) is not mathematically rigorous; the sifting property give a definition that is. *For any function $x(t)$ that is finite and continuous at $t = t_0$, when multiplied by the unit impulse $\delta(t - t_0)$, and the product is integrated between limits which include $t = t_0$, the result is*

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\,dt = x(t_0) \tag{2.45}$$

One can argue here, what if we can find a function other than the impulse function that when multiplied with any function $x(t)$ that is continuous and finite at $t = t_0$ and the product is integrated satisfies the same result in (2.45). The answer would be that this function must be equivalent to the impulse function. Next we show that the derivative of the unit step function is equivalent to the unit impulse.

### The Unit Impulse and its relation to the Unit Step

Consider a function $x(t)$ and its derivative $\frac{dx(t)}{dt}$ as in Figure 2.37. In the limit as $a$ approaches zero the function $x(t)$ approaches the unit step function. In that same limit the width of $\frac{dx(t)}{dt}$ approaches zero but maintains a unit area which is the same as the initial definition of $\delta_a(t)$. The limit as $a$ approaches zero of $\frac{dx(t)}{dt}$ is called the *generalised derivative* of $u(t)$.

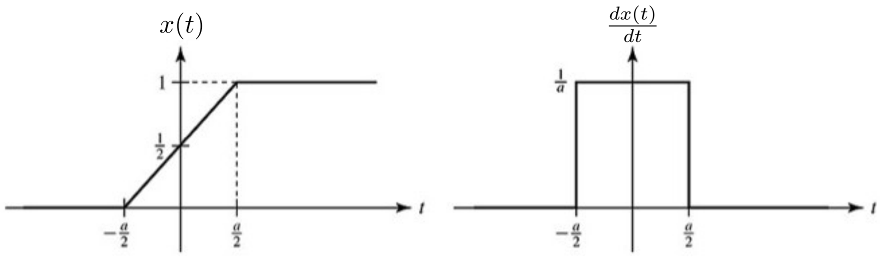



**Figure 2.37:** Functions which approach the unit step and unit impulse

Since the unit impulse is the generalized derivative of the unit step, it must follow that the unit step is the integral of the unit impulse,

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau \tag{2.46}$$

The result in (2.46) can be obtained by observing that the area from $-\infty$ to $t$ is zero if $t < 0$, because the unit impulse is not included in the integration range and unity if $t > 0$ since the integral of the unit impulse whose integration range includes $t = 0$ must have the value of one

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

**Example 2.13** Show that the generalized derivative of $u(t)$ satisfies the sifting property.

■ **Solution** Let us evaluate the integral $(du/dt)x(t)$, using integration by parts:

$$\int_{-\infty}^{\infty} \frac{du(t)}{dt} x(t)\, dt = u(t)x(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\dot{x}(t)\, dt$$
$$= x(\infty) - 0 - \int_{0}^{\infty} \dot{x}(t)\, dt$$
$$= x(\infty) - x(t)\Big|_{0}^{\infty} = x(0)$$

The result shows that $du/dt$ satisfies the sifting property of $\delta(t)$, i.e., the generalized derivative of the unit step extracts the value of any continuous finite function $x(t)$ at time $t = 0$ when the multiplication of $\frac{d}{dt}u(t)$ and $x(t)$ is integrated between any two limits which include time $t = 0$. Therefore,

$$\frac{d}{dt}u(t) = \delta(t) \quad ■ \tag{2.47}$$

THE SCALING PROPERTY

The important feature of the unit impulse function is not its shape but the fact that its width approaches zero while the area remains at unity. Therefore, when time transformations are applied to $\delta(t)$, in particular scaling it is the strength that matters and not the shape of $\delta(t)$, (Figure 2.38). It is helpful to note that

$$\int_{-\infty}^{\infty} \delta(\alpha t) dt = \int_{-\infty}^{\infty} \delta(\lambda) \frac{d\lambda}{|\alpha|} = \frac{1}{|\alpha|} \tag{2.48}$$

and so

$$\delta(\alpha t) = \frac{\delta(t)}{|\alpha|} \tag{2.49}$$

In general, it can be shown that

$$\delta(\alpha t - \beta) = \frac{1}{|\alpha|} \delta\left(t - \frac{\beta}{\alpha}\right) \tag{2.50}$$

**Example 2.14** Sketch the following functions: $\delta(3t)$, $\delta\left(\frac{t-1}{2}\right)$, and $\delta\left(\frac{t}{2} - 1\right)$.

■ **Solution** Using (2.50), we obtain the sketchs shown in Figure 2.38. ■

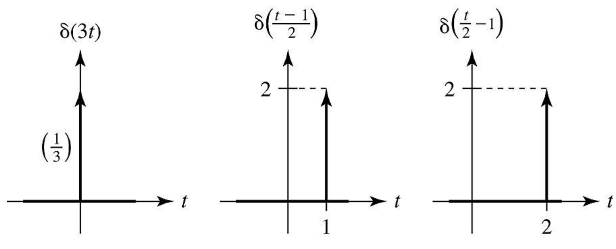



**Figure 2.38:** Effect of scaling on unit impulse

**Example 2.15**

Evaluate the following integrals:

a. $\int_{-2}^{4} (t + t^2)\delta(t - 3)\, dt$

b. $\int_{0}^{3} e^{(t-2)}\delta(2t - 4)\, dt$

■ **Solution**

a. Using the sifting property yields

$$\int_{-2}^{4} (t + t^2)\delta(t - 3)\, dt = 3 + 3^2 = 12$$

since $t = 3$ is within the integration range. Note, that if the upper limit of the integration was one the result would have been zero since $t = 3$ will not be in the integration range.

b. Using the scaling property then the sifting property yields

$$\begin{aligned}\int_{0}^{3} e^{(t-2)}\delta(2t - 4)\, dt &= \int_{0}^{3} e^{(t-2)}\tfrac{1}{2}\delta(t - 2)\, dt \\ &= \tfrac{1}{2}e^0 = \tfrac{1}{2} \quad ■\end{aligned}$$

Table 2.5 lists the definition and several properties of the unit impulse function. The properties listed in Table 2.5 are very useful in the signal and system analysis.

**Table 2.5:** Properties of the Unit Impulse Function

| | |
|---|---|
| 1. | $\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0)$, $x(t)$ continuous at $t = t_0$ |
| 2. | $\int_{t_1}^{t_2} x(t)\delta(t - t_0)\, dt = \begin{cases} x(t_0), & t_1 < t_0 < t_2 \\ 0, & \text{otherwise} \end{cases}$ |
| 3. | $\int_{-\infty}^{\infty} x(t - t_0)\delta(t)\, dt = x(-t_0)$, $x(t)$ continuous at $t = -t_0$ |
| 4. | $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$, $x(t)$ continuous at $t = t_0$ |
| 5. | $\delta(t - t_0) = \frac{d}{dt}u(t - t_0)$ |
| 6. | $u(t - t_0) = \int_{-\infty}^{t} \delta(\tau - t_0)d\tau = \begin{cases} 1, & t > t_0 \\ 0, & t < t_0 \end{cases}$ |
| 7. | $\delta(\alpha t - \beta) = \frac{1}{\lvert\alpha\rvert}\delta\left(t - \frac{\beta}{\alpha}\right)$ |
| 8. | $\int_{t_1}^{t_2} \delta(\alpha t - \beta)\, dt = \begin{cases} \frac{1}{\lvert\alpha\rvert}\int_{t_1}^{t_2} \delta\left(t - \frac{\beta}{\alpha}\right)\, dt, & t_1 < \frac{\beta}{\alpha} < t_2 \\ 0, & \text{otherwise} \end{cases}$ |
| 9. | $\delta(t) = \delta(-t)$ |

### The DT Unit Impulse Function

The DT unit impulse function $\delta[n]$, sometimes referred to as the Kronecker delta function is defined by

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{2.51}$$

$$\delta[n-k] = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$$

and is shown in Figure 2.39. The DT delta function $\delta[n]$ is referred to as the unit sample that occurs at $n = 0$ and the shifted function $\delta[n-k]$ as the sample that occurs at $n = k$ .

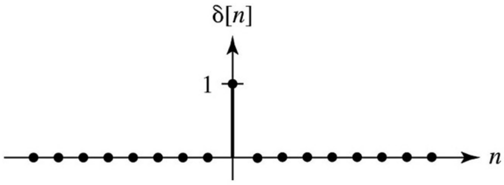



**Figure 2.39:** The DT unit impulse function.

### Some properties of $\delta[n]$

1. $\delta[n] = 0$ for $n \neq 0$.

2. $\sum_{m=-\infty}^{n} \delta[m] = u[n]$, this can be easily seen by considering two cases for $n$, namely $n < 0$ and $n > 0$

   - *Case* 1: $\sum_{m=-\infty}^{n} \delta[m] = 0$ for $n < 0$, this is true since $\delta[m]$ has a value of one only when $m = 0$ and zero anywhere else. The upper limit of the summation is less than zero thus $\delta[m = 0]$ is not included in the summation.
   - *Case* 2: On the other hand if $n \geq 0$, $\delta[m = 0]$ will be included in the summation, therefore $\sum_{m=-\infty}^{n} \delta[m] = 1$.

   In summary,

$$\sum_{m=-\infty}^{n} \delta[m] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$
$$= u[n]$$

3. $u[n] - u[n-1] = \delta[n]$, this can be clearly see in Figure 2.40 as you subtract the two signals from each other.

4. $\sum_{k=0}^{\infty} \delta[n-k] = u[n]$.

5. $x[n]\delta[n] = x[0]\delta[n]$.

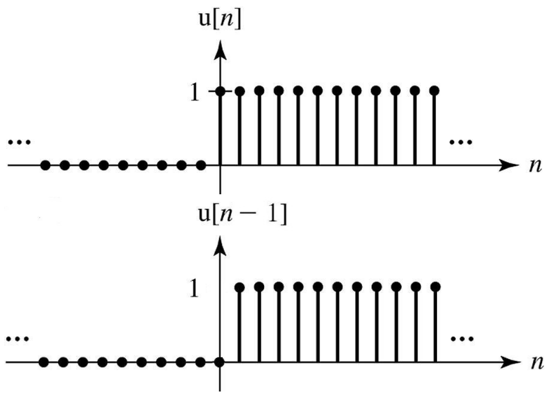



**Figure 2.40:** $\delta[n] = u[n] - u[n-1]$

6. $x[n]\delta[n-k] = x[k]\delta[n-k]$.
7. The DT unit impulse is not affected by scaling, i.e. $\delta[\alpha n] = \delta[n]$.
8. I will leave out some other important properties to a later stage of this course in particular the sifting property of the DT unit impulse.

Table 2.6 lists the equivalent properties of both the CT and the DT.

**Table 2.6:** Equivalent operations

| | **Continuous Time** | **Discrete Time** |
|---|---|---|
| 1. | $\int_{-\infty}^{t} x(\tau)\, d\tau$ | $\sum_{-\infty}^{n} x[k]$ |
| 2. | $\frac{dx(t)}{dt}$ | $x[n] - x[n-1]$ |
| 3. | $x(t)\delta(t) = x(0)\delta(t)$ | $x[n]\delta[n] = x[0]\delta[n]$ |
| 4. | $\delta(t) = \frac{du(t)}{dt}$ | $\delta[n] = u[n] - u[n-1]$ |
| 5. | $u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$ | $u[n] = \sum_{k=-\infty}^{n} \delta[k]$ |

### 2.3.8 The Unit Sinc Function

The unit sinc function (Figure 2.41) is called a *unit* function because its height and area are both one, it is defined as

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{2.52}$$

Some authors define the sinc function as

$$\mathrm{Sa}(t) = \frac{\sin(t)}{t} \tag{2.53}$$

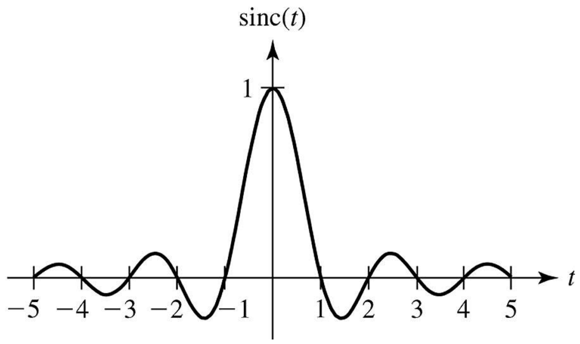



**Figure 2.41:** The CT unit sinc function

and is more commonly known as the sampling function $\mathrm{Sa}(t)$. Note that $\mathrm{sinc}(t) = \mathrm{Sa}(\pi t)$. One can use either of them as long as one definition is used consistently. What is sinc(0)? To determine the value of sinc(0), simply use L'Hôpital's rule to the definition in (2.52). Then

$$\lim_{t\to 0} \mathrm{sinc}(t) = \lim_{t\to 0} \frac{\sin(\pi t)}{\pi t} = \lim_{t\to 0} \frac{\pi \cos(\pi t)}{\pi} = 1.$$

Chapter 3

# Description of Systems

## 3.1 Introduction

The words signal and systems were defined very generally in Chapter 1. Systems can be viewed as any process or interaction of operations that transforms an input signal into an output signal with properties different from those of the input signals. A system may consist of physical components (hardware realization) or may consist of an algorithm that computes the output signal from the input signal (software realization). One way to define a system is anything that performs a function, it operates on something to produce something else. It can be thought of as a mathematical operator. A CT system operates on a CT input signal to produce a CT output. The system may be denoted

$$y(t) = \mathcal{H}\{x(t)\} \tag{3.1}$$

$\mathcal{H}$ is the operator denoting the action of a system, it specifies the operation or transformation to be performed and also identifies the system. On the other hand, a DT system operates on a DT signal to produce a DT output, (Figure 3.1. Occasionally, we use the following notation to describe a system

$$x(t) \xrightarrow{\mathcal{H}} y(t)$$

which simply means the input $x$ to system $\mathcal{H}$ produces the output $y$. The set of equations relating the input $x(t)$ and the output $y(t)$ is called the *mathematical model*, or simply, the model, of the system. Given the input $x(t)$, this set of

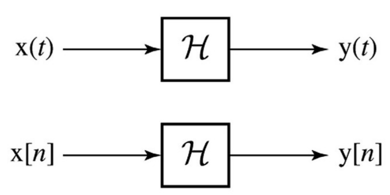



**Figure 3.1:** CT and DT system block diagrams.

equations must be solved to obtain $y(t)$. For CT system, the model is usually a set of differential equations. As an example of a very simple system which is familiar to electrical engineers is a circuit. Circuits are electrical systems. A very common circuit is the $RC$ lowpass filter, a single-input, single-output system, illustrated in Figure 3.2. The voltage at the input $v_{in}(t)$ is the excitation of the system, and the voltage at the output $v_{out}(t)$ is the response of the system. This system consists of two components, a resistor and a capacitor. The mathematical voltage-current relations for resistors and capacitors are well known and are illustrated in Figure 3.3. By knowing how to mathematically describe and characterize all the components in a system and how the components interact with each other, an engineer can predict, using mathematics, how a system will work, without actually building it.

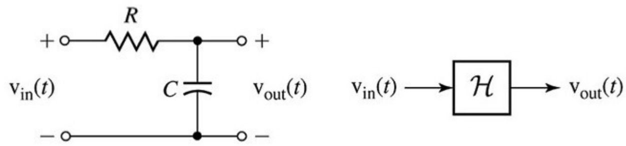



**Figure 3.2:** An $RC$ lowpass filter: a SISO system.

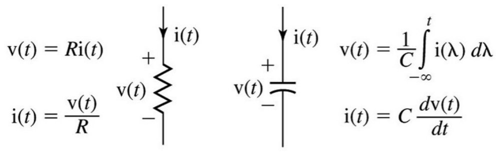



**Figure 3.3:** Mathematical voltage-current relationships for a resistor and a capacitor.

INTERCONNECTING SYSTEMS

A system is often described and analyzed as an assembly of components. The study of systems is the study of how interconnected components function as a whole. Using block diagrams to describe different system components is very convenient. The block shown in Figure 3.1 is a graphical representation of a system described by (3.1). Figure 3.4 illustrates the basic block diagram elements we mostly use in this course. There are some common ways that

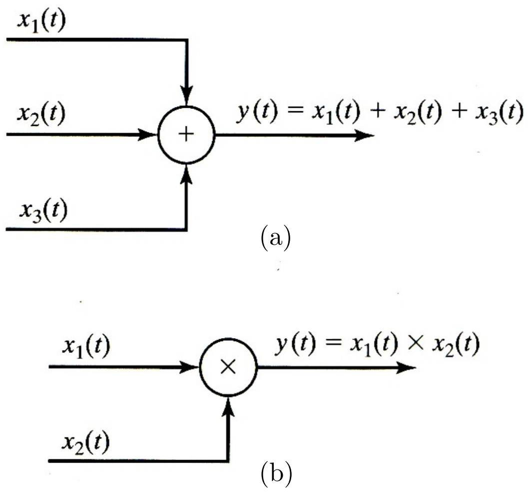
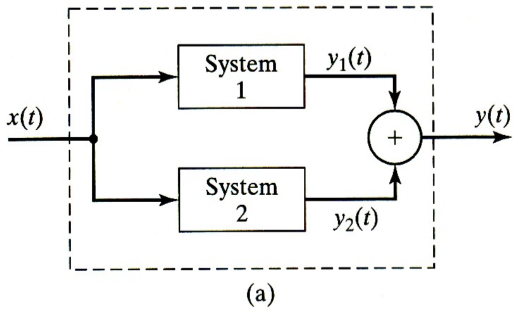



**Figure 3.4:** (a) Summing junction (b) Product junction.

systems are connected to form larger systems. Two of the most common are



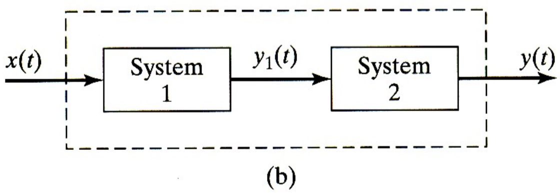




**Figure 3.5:** (a) Parallel connection of systems (b) Cascaded connection of systems.

the cascade connection and the parallel connection as illustrated in Figure 3.5.

## 3.2 Systems Characteristics

in this section, we define some of the properties, or characteristics, of systems. These definitions apply to both CT or DT systems. Systems may be classified to the following categories:

1. Memoryless (instantaneous) and dynamic (with memory) systems.
2. Invertible and non-invertible systems.
3. Causal and non-causal systems.
4. Stable and non-stable systems.
5. Time-invariant and time-varying systems.
6. Linear and non-linear systems.

### 3.2.1 Memory

A systems output or response at any instant $t$ generally depends upon the entire past input. However, there are systems for which the output at any instant $t$ depends only on its input at that instant and not on any other input at any other time. Such systems are said to have no memory or is called *memoryless.* The only input contributing to the output of the system occurs at the same time as the output. The system has no stored information of any past inputs thus the term memoryless. Such systems are also called *static* or *instantaneous*

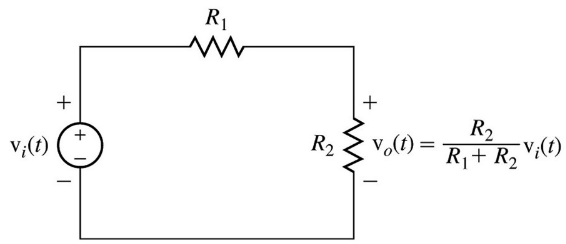



**Figure 3.6:** A voltage divider.

*systems.* Otherwise, the system is said to be *dynamic* (or a system with memory). Instantaneous systems are a special case of dynamic systems. An example of memoryless system is the voltage divider circuit shown in Figure 3.6.

As an example of a system with memory is a capacitor, the voltage-current relationship is defined as:

$$v(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau)\, d\tau$$

The voltage across the capacitor at time $t_0$ depends on the current $i(t)$ for all time before $t_0$. Thus the system has memory.

### 3.2.2 Invertibility

A system $\mathcal{H}$ performs certain operations on input signals. If we can obtain the input $x(t)$ back from the output $y(t)$ by some operation, the system $\mathcal{H}$ is said to be *invertible*. Thus, an inverse system $\mathcal{H}^{-1}$ can be created so that when the output signal is fed into it, the input signal can be recovered (Figure 3.7). For a non-invertible system, different inputs can result in the same output, and it is impossible to determine the input for a given output. Therefore, for an invertible system it is essential that distinct inputs applied to the system produce distinct outputs so that there is one-to-one mapping between an input and the corresponding output. An example of a system that is not invertible is a system that performs the operation of squaring the input signals, $y(t) = x^2(t)$. For any given input $x(t)$ it is possible to determine the value of the output $y(t)$. However, if we attempt to find the output, given the input, by rearranging the relationship into $x(t) = \sqrt{y(t)}$ we face a problem. The square root function has multiple values, for example $\sqrt{4} = \pm 2$. Therefore, there is no one to one mapping between an input and the corresponding output signals. In other words we have the same output for different inputs. An example of a system that is

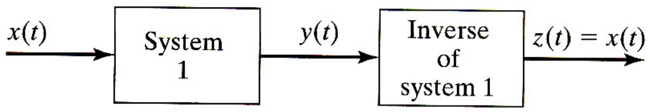



**Figure 3.7:** The inverse system

invertible, consider an inductor whose input-output relationship is described by

$$i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau)\, d\tau$$

the operation representing the inverse system is simply: $L\dfrac{d}{dt}$.

### 3.2.3 Causality

A *causal* system is one for which the output at any instant $t_0$ depends only on the value of the input $x(t)$ for $t \leq t_0$. In other words, the value of the current output depends only on current and past inputs. This should seem obvious as how could a system respond to an input signal that has not yet been applied. Simply, the output cannot start before the input is applied. A system that violates the condition of causality is called a *noncausal* system. A noncausal system is also called *anticipative* which means the systems knows the input in the future and acts on this knowledge before the input is applied. Noncausal systems do not exist in reality as we do not know yet how to build a system that can respond to inputs not yet applied. As an example consider the system specified by $y(t) = x(t+1)$. Thus, if we apply an input starting at $t = 0$, the output would begin at $t = -1$, as seen in Figure 3.8 hence a noncausal system. On the other hand a system described by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

is clearly a causal system since the output $y(t)$ depends on inputs that occur since $-\infty$ up to time $t$ (the upper limit of the integral). If the upper limit is given as $t+1$ the system is noncausal.

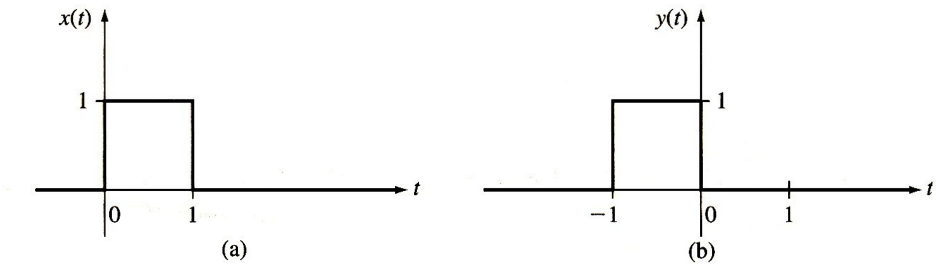



**Figure 3.8:** A noncausal system

### 3.2.4 Stability

A system is *stable* if a bounded input signal yields a bounded output signal. A signal is said bounded if its absolute value is less than some finite value for all time,

$$|x(t)| < \infty, \qquad -\infty < t < \infty.$$

A system for which the output signal is bounded when the input signal is bounded is called *bounded-input-bounded-output* (BIBO) stable system. Bounded $x(t)$ and $y(t)$ are illustrated in Figure 3.9.

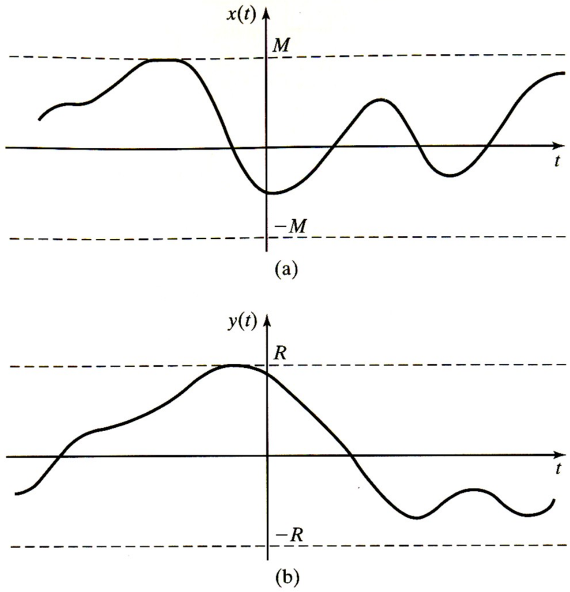



**Figure 3.9:** Bounded functions

### 3.2.5 Time Invariance

A system is *time-invariant* if the input-output properties do not change with time. For such a system, if the input is delayed by $t_0$ seconds, the output is the same as before but delayed by $t_0$ seconds. In other words, a time shift in the input signal causes the same time shift in the output signal without changing the functional form of the output signal. If input $x(t)$ yields output $y(t)$ the input $x(t-t_0)$ yields output $y(t-t_0)$ for all $t_0 \in \mathcal{R}$, i.e.

$$x(t) \xrightarrow{\mathcal{H}} y(t) \Longrightarrow x(t-t_0) \xrightarrow{\mathcal{H}} y(t-t_0).$$

A system that is not time invariant is *time varying*. A test for time invariance is illustrated in Figure 3.10. The signal $y(t-t_0)$ is obtained by delaying $y(t)$ by $t_0$. We define $y_d(t)$ as the system output for the delayed input $x(t-t_0)$. The system is time invariant, provided that $y_d(t) = y(t-t_0)$.

**Example 3.1** Determine which of the following systems is time-inariant:

(a) $y(t) = \cos x(t)$

(b) $y(t) = x(t) \cos t$

■ **Solution** Consider the system in part (a). Using the test procedure for time invariance illustrated in Figure 3.10

$$y_d(t) = y(t)\Big|_{x(t-t_0)} = \cos x(t-t_0) = y(t)\Big|_{t-t_0}$$

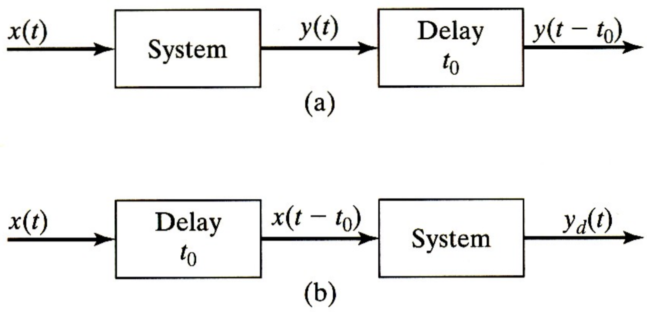



**Figure 3.10:** Test for time invariance.

and the system is time invariant. Now consider the system in part (b)

$$y_d(t) = y(t)\Big|_{x(t-t_0)} = x(t - t_0)\cos t$$

and

$$y(t)\Big|_{t-t_0} = x(t - t_0)\cos(t - t_0)$$

Comparison of the last two expressions leads to the conclusion that the system is time varying. It is easier to do the test on a block diagram representation as illustrated in Figure 3.11. Note that $y(t - t_0)$ in Figure 3.11(a) and $y_d(t)$ in Figure 3.11(b) are not equal, therefore the system in time varying. ■

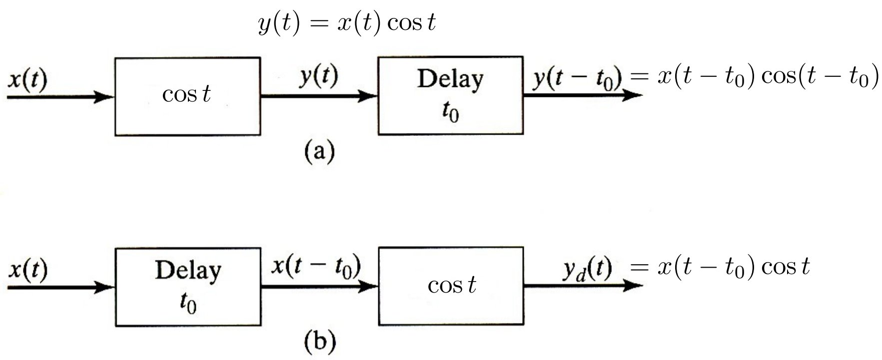



**Figure 3.11:** Test for time invariance of the system $y(t) = x(t)\cos t$.

### 3.2.6 Linearity and Superposition

#### Homogeneity (Scaling) Property

A system is said to be *homogenous* for arbitrary real or complex number $K$ if the input signal is increased $K$-fold, the output signal also increases $K$-fold.

Thus, if

$$x(t) \xrightarrow{\mathcal{H}} y(t)$$

then for all real or imaginary $K$

$$Kx(t) \xrightarrow{\mathcal{H}} Ky(t)$$

Figure 3.12 illustrates, in block diagram representation, what homogeneity means.

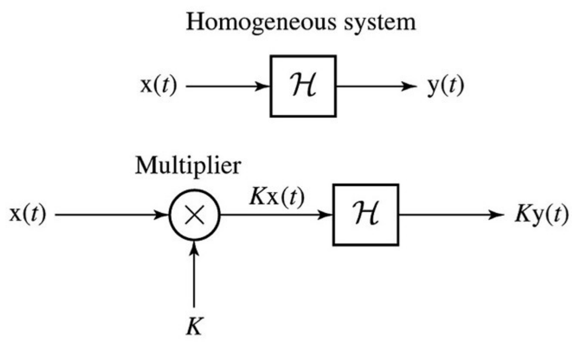



**Figure 3.12:** Block diagram illustrating the concept of homogeneity ($K$ is any complex constant.

Additivity Property

The additivity property of a system implies that if several inputs are acting on the system, then the total output of the system can be determined by considering each input separately while assuming all the other inputs to be zero. The total output is then the sum of all the component outputs. This property may be expressed as follows: if an input $x_1(t)$ acting alone produces an output $y_1(t)$, and if another input $x_2(t)$, also acting alone, has an output $y_2(t)$, then, with both inputs acting together on the system, the total output will be $y_1(t)+y_2(t)$. Thus, if

$$x_1(t) \xrightarrow{\mathcal{H}} y_1(t) \quad \text{and} \quad x_2(t) \xrightarrow{\mathcal{H}} y_2(t)$$

then

$$x_1(t) + x_2(t) \xrightarrow{\mathcal{H}} y_1(t) + y_2(t).$$

The block diagram in Figure 3.13 illustrates the concept of additivity.

A system is *linear* if both the homogeneity and the additivity property are satisfied. Both these properties can be combined into one property (*superposition*) which can be expressed as follows: if

$$x_1(t) \xrightarrow{\mathcal{H}} y_1(t) \quad \text{and} \quad x_2(t) \xrightarrow{\mathcal{H}} y_2(t)$$

then for all real or imaginary $\alpha$ and $\beta$,

$$\alpha x_1(t) + \beta x_2(t) \xrightarrow{\mathcal{H}} \alpha y_1(t) + \beta y_2(t).$$

**Example 3.2**

Determine whether the system described by the differential equation

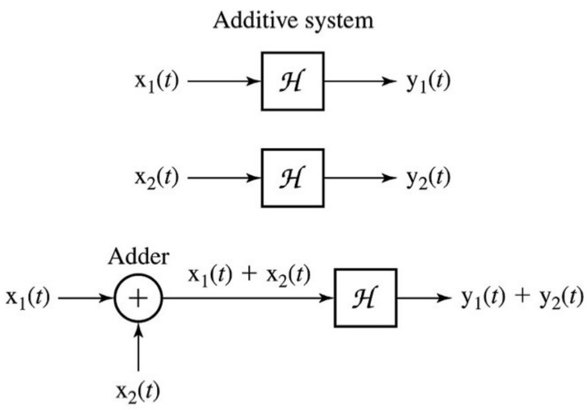



**Figure 3.13:** Block diagram illustrating the concept of additivity.

$$a\ddot{y}(t) + by^2(t) = x(t)$$

is linear or nonlinear.

■ **Solution** Consider two individual inputs $x_1(t)$ and $x_2(t)$, the equations describing the system for the two inputs acting alone would be

$$a\ddot{y}_1(t) + by_1^2(t) = x_1(t) \quad \text{and} \quad a\ddot{y}_2(t) + by_2^2(t) = x_2(t)$$

The sum of the two equations is

$$a[\ddot{y}_1(t) + \ddot{y}_2(t)] + b[y_1^2(t) + y_2^2(t)] = x_1(t) + x_2(t)$$

which is not equal to

$$a[y_1(t) + y_2(t)]'' + b[y_1(t) + y_2(t)]^2 = x_1(t) + x_2(t).$$

Therefore, in this system superposition is not applied hence the system is nonlinear. ■

**Remark** For a system to be linear a zero input signal implies a zero output. Consider for an example the system

$$y[n] = 2x[n] + x_0$$

where $x_0$ might be some initial condition or a dc component. If $x[n] = 0$ it is clear that $y[n] \neq 0$ which is not linear unless $x_0$ is zero.

## 3.3 Linear Time-invariant Systems

In this course we are involved in the analysis of linear time-invariant (LTI) systems. Many engineering systems are well approximated by LTI models, analysis of such systems is simple and elegant. We consider two methods of analysis of LTI systems: the time-domain method and the frequency-domain method. Some of the frequency domain methods will be addressed later in the course.

### 3.3.1 Time-Domain Analysis of LTI systems

Analysis for DT systems will be introduced first, as it is easier to analyze, it will then be extended to CT systems. Recall that by analysis we mean determining the response $y[n]$ of a LTI system to an arbitrary input $x[n]$.

#### Unit Impulse Response $h[n]$

The unit impulse function, $\delta[n]$, is used extensively in determining the response of a DT LTI system. When the input signal to the system is $\delta[n]$ the output is called the *impulse response*, $h[n]$

$$\delta[n] \xrightarrow{\mathcal{H}} h[n]$$

If we know the system response to an impulse input, and if an arbitrary input $x[n]$ can be expressed as a sum of impulse components, the system response could be obtained by summing the system response to various impulse components.

#### Impulse representation of DT signals

Here, we show that an arbitrary signal $x[n]$ can be expressed as a function of impulse functions. Recall the definition of the DT impulse function (also called the unit sample function):

$$\delta[n-k] = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases} \tag{3.2}$$

An impulse function has a value of unity when its argument is zero; otherwise, its value is zero. From this definition we see that

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

Consider the signal $x[n]$ shown in Figure 3.14(a). Using (3.2), we can see that

$$x[n]\delta[n+1] = x[-1]\delta[n+1] = \begin{cases} x[-1], & n = -1 \\ 0, & n \neq -1 \end{cases}$$

In a like manner,

$$x[n]\delta[n] = x[0]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n \neq 0 \end{cases}$$

and

$$x[n]\delta[n-1] = x[1]\delta[n-1] = \begin{cases} x[1], & n = 1 \\ 0, & n \neq 1 \end{cases}$$

Hence, we can express the signal $x[n]$ as

$$\begin{aligned} x[n] &= x[-1] + x[0] + x[1] \\ &= x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] \\ &= \sum_{k=-1}^{1} x[k]\delta[n-k] \end{aligned}$$

In general the component of $x[n]$ at $n = k$ is $x[k]\delta[n-k]$, and $x[n]$ is the sum of all these components summed from $k = -\infty$ to $\infty$. Therefore,

$$\begin{aligned} x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots \\ + x[-1]\delta[n+1] + x[-2]\delta[n+2] + \cdots \\ = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \end{aligned} \tag{3.3}$$

The expression in (3.3) is the DT version of the *sifting property,* $x[n]$ is written as a weighted sum of unit impulses.

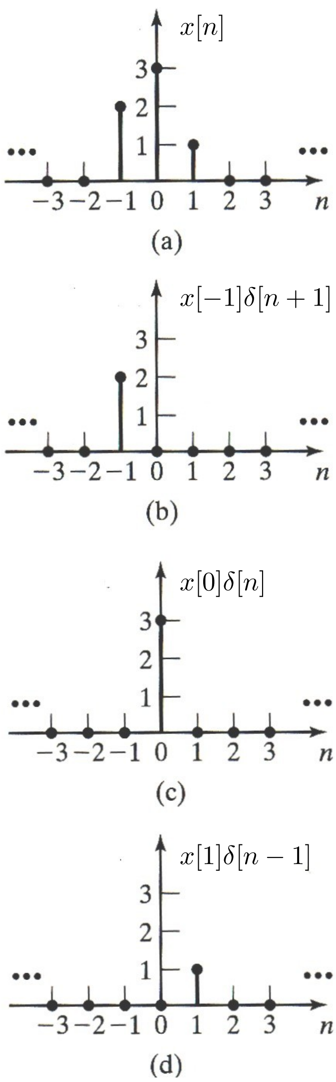



**Figure 3.14:** Representation of an arbitrary signal $x[n]$ in terms of impulse components

**Example 3.3** Express the signal shown in Figure 3.15 as a weighted sum of impulse components.

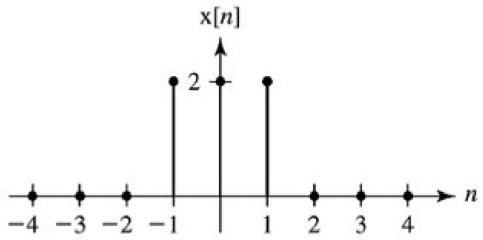



**Figure 3.15:** A DT signal $x[n]$.

■ **Solution** Using (3.3), it can be easily shown that

$$x[n] = 2\delta[n+1] + 2\delta[n] + 2\delta[n-1] \quad ■$$

### 3.3.2 The Convolution Sum

We are interested in finding the system output $y[n]$ to an arbitrary input $x[n]$ knowing the impulse response $h[n]$ to a DT LTI system. There is a systematic way of finding how the output responds to an input signal, it is called *convolution.* The convolution technique is based on a very simple idea, no matter how complicated your input signal is, one can always express it in terms of weighted impulse components. For LTI systems we can find the response of the system to one impulse component at a time and then add all those responses to form the total system response. Let $h[n]$ be the system response (output) to impulse input $\delta[n]$. Thus if

$$\delta[n] \xrightarrow{\mathcal{H}} h[n]$$

then because the system is time-invariance

$$\delta[n-k] \xrightarrow{\mathcal{H}} h[n-k]$$

and because of linearity, if the input is multiplied by a weight or constant the output is multiplied by the same weight thus

$$x[k]\delta[n-k] \xrightarrow{\mathcal{H}} x[k]h[n-k]$$

and again because of linearity

$$\underbrace{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}_{x[n]} \xrightarrow{\mathcal{H}} \underbrace{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{y[n]}$$

The left hand side is simply $x[n]$ [see equation (3.3)], and the right hand side is the system response $y[n]$ to input $x[n]$. Therefore

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{3.4}$$

The summation on the RHS is known as the *convolution sum* and is denoted by $y[n] = x[n] * h[n]$. Now in order to construct the response or output of a DT LTI system to any input $x[n]$, all we need to know is the system's impulse response $h[n]$. Hence, the impulse response $h[n]$ of a discrete LTI system contains a complete *input-output description* of the system.
We shall evaluate the convolution sum first by analytical method and later with graphical aid.

**Example 3.4**

Determine $y[n] = x[n] * h[n]$ for $x[n]$ and $h[n]$ as shown in Figure 3.16

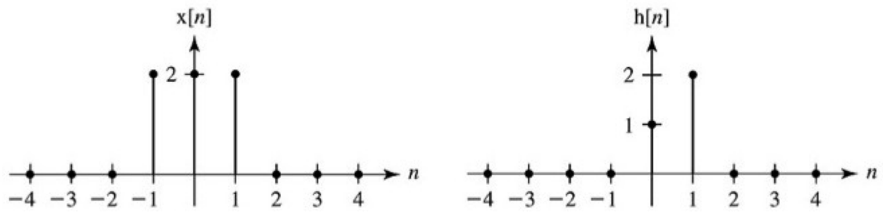



**Figure 3.16:** Two DT signals $x[n]$ and $h[n]$.

■ **Solution** (METHOD 1) Express $x[n]$ as weighted sum of impulse components

$$x[n] = 2\delta[n+1] + 2\delta[n] + 2\delta[n-1]$$

Since the system is an LTI one, the output is simply the summation of the impulse responses to individual components of $x[n]$, therefore,

$$y[n] = 2h[n+1] + 2h[n] + 2h[n-1] \tag{3.5}$$

Impulse responses to individual components of $x[n]$ are illustrated in Figure 3.17 and $y[n]$ is shown in Figure 3.18.

(METHOD 2) By direct evaluation of the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

which can be written as

$$\begin{aligned} y[n] = \cdots + x[-2]h[n+2] + x[-1]h[n+1] + x[0]h[n] \\ + x[1]h[n-1] + x[2]h[n-2] + \cdots \end{aligned}$$

and for $x[n]$ in Figure 3.16 we have

$$\begin{aligned} y[n] &= x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] \\ &= 2h[n+1] + 2h[n] + 2h[n-1] \end{aligned} \tag{3.6}$$

which is the same as equation (3.5). We can now graph $y[n]$ as illustrated in Figure 3.18. ■

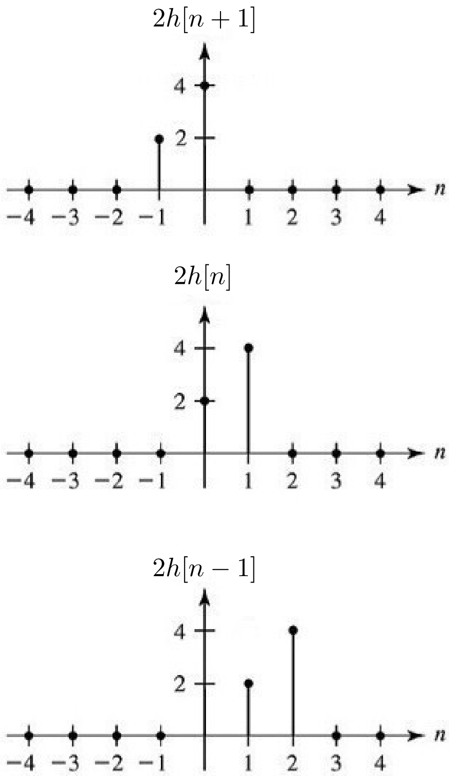



**Figure 3.17:** Impulse response to individual components of $x[n]$.

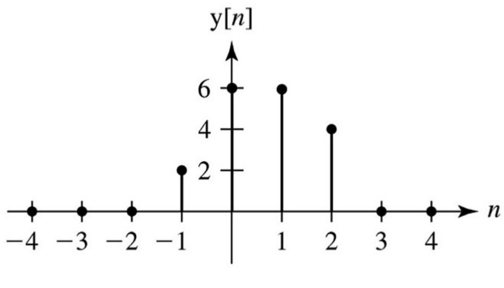



**Figure 3.18:** Graph of $y[n]$

PROPERTIES OF THE CONVOLUTION SUM

1. THE COMMUTATIVE PROPERTY

$$x[n] * h[n] = h[n] * x[n] \tag{3.7}$$

This property can be easily proven by starting with the definition of convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

and letting $q = n - k$. Then we have

$$x[n] * h[n] = \sum_{q=-\infty}^{\infty} x[n-q]h[q] = \sum_{q=-\infty}^{\infty} h[q]x[n-q] = h[n] * x[n]$$

This property is illustrated in Figure 3.19, the output for each system is identical.

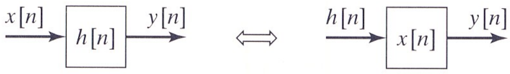



**Figure 3.19:** Commutative property.

2. THE DISTRIBUTIVE PROPERTY

$$x[n] * (h[n] + z[n]) = x[n] * h[n] + x[n] * z[n] \tag{3.8}$$

If we convolve $x[n]$ with the sum of $h[n]$ and $z[n]$, we get

$$\begin{aligned} x[n] * (h[n] + z[n]) &= \sum_{k=-\infty}^{\infty} x[k]\left(h[n-k] + z[n-k]\right) \\ &= \underbrace{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{=x[n]*h[n]} + \underbrace{\sum_{k=-\infty}^{\infty} x[k]z[n-k]}_{=x[n]*z[n]} \end{aligned}$$

This property is illustrated by two systems in parallel as in Figure 3.20, where the output is given by

$$y[n] = x[n] * h_1[n] + x[n] * h_2[n] = x[n] * (h_1[n] + h_2[n])$$

Therefore, the total system impulse response is the sum of the impulse responses:

$$h[n] = h_1[n] + h_2[n]$$

This can be extended to a parallel connection of any number of systems. The impulse response of a parallel connection of LTI system is the sum of all the individual system impulse responses.

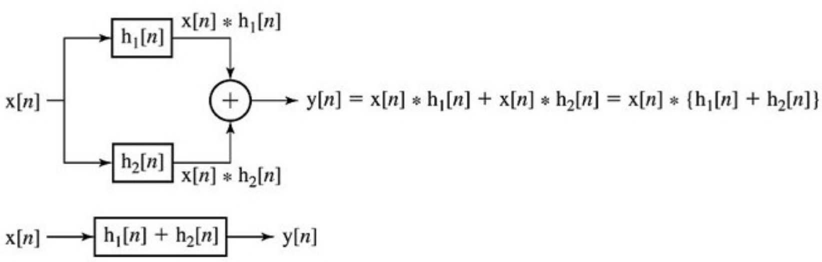



**Figure 3.20:** Distributive property.

3. The Associative Property

$$x[n] * (h[n] * z[n]) = (x[n] * h[n]) * z[n] \tag{3.9}$$

The proof to this property is left as an exercise to the reader. As an example of this property, consider the output of the system of Figure 3.21, which is given by

$$y[n] = y_1[n] * h_2[n] = (x[n] * h_1[n]) * h_2[n]$$

Then using (3.9),

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]) = x[n] * (h_2[n] * h_1[n])$$

Hence, the order of the two systems of Figure 3.21(a) may be replaced with a single system with the impulse response

$$h[n] = h_1[n] * h_2[n]$$

such that the input-output characteristics remain the same. This property is illustrated in Figure 3.21(b).

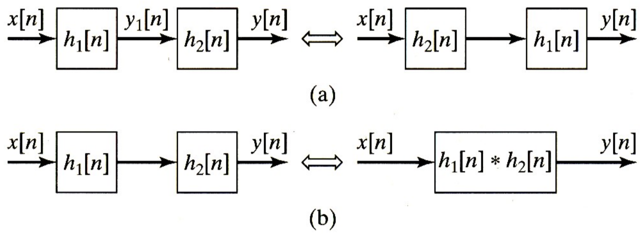



**Figure 3.21:** Associative property.

4. The Shifting property

$$x[n-m] * h[n-q] = y[n-m-q] \tag{3.10}$$

In words, the input $x$ is delayed by $m$ samples, the signal $h$ is also delayed by $q$ samples, therefore the result of the convolution of both signals will introduce a total delay in the output signal by $m+q$ samples.

5. Convolution with an Impulse

$$x[n] * \delta[n] = x[n] \tag{3.11}$$

This property can be easily seen from the definition of convolution

$$x[n] * \delta[n[= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{3.12}$$

and the RHS in (3.12) is simply $x[n]$ from (3.3).

6. The Width Property

   If $x[n]$ and $h[n]$ have lengths of $m$ and $n$ elements respectively, then the length of $y[n]$ is $m + n - 1$ elements. *In some special cases this property could be violated.* One should be careful to count samples with zero amplitudes that exist in between the samples. Furthermore, the appearance of the first sample in the output will be located at the summation of the locations of the first appearing samples of each function. This also applies to the last appearing sample.

   To demonstrate this property recall from Example 3.4, $x[n]$ and $h[n]$ had a width of three samples and two samples respectively. The first sample in $x[n]$ appeared at $n = -1$, and in $h[n]$ at $n = 0$. Therefore, we would expect $y[n]$ to have a width of $(3 + 2 - 1 = 4)$ samples. Furthermore, we would expect the first sample in the output $y[n]$ to appear at $n = -1 + 0 = -1$ and the last sample at $n = 1 + 1 = 2$. Figure 3.18 clearly demonstrate these expectations.

### Graphical Procedure for the Convolution Sum

The direct analytical methods to evaluate the convolution sum are simple and convenient to use as long as the number of samples are small. It is helpful to explore some graphical concepts that helps in performing convolution of more complicated signals. If $y[n]$ is the convolution of $x[n]$ and $h[n]$, then

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{3.13}$$

It is crucial to note that the summation index in (3.13) is $k$, so that $n$ is just like a constant. With this in mind $h[n-k]$ should be considered a function of $k$ for purposes of performing the summation in (3.13). This consideration is also important when we sketch the graphical representations of the functions $x[k]$ and $h[n-k]$. Both of these functions should be sketched as functions of $k$, not of $n$. To understand what the function $h[n-k]$ looks like let us start with the function $h[k]$ and perform the following transformations

$$h[k] \xrightarrow{k \to -k} h[-k] \xrightarrow{k \to k-n} h[-(k-n)] = h[n-k]$$

The first transformation is a time reflected version of $h[k]$, and the second transformation shifts the already reflected function $n$ units to the right for positive $n$; for negative $n$, the shift is to the left as illustrated in Figure 3.22. The convolution operation can be performed as follows:

1. Reflect $h[k]$ about the vertical axis ($n = 0$) to obtain $h[-k]$.

2. Time shift $h[-k]$ by $n$ units to obtain $h[n-k]$. For $n > 0$, the shift is to the right; for $n < 0$, the shift is to the left.

3. Multiply $x[k]$ by $h[n-k]$ and add all the products to obtain $y[n]$. The procedure is repeated for each value $n$ over the range $-\infty$ to $\infty$.

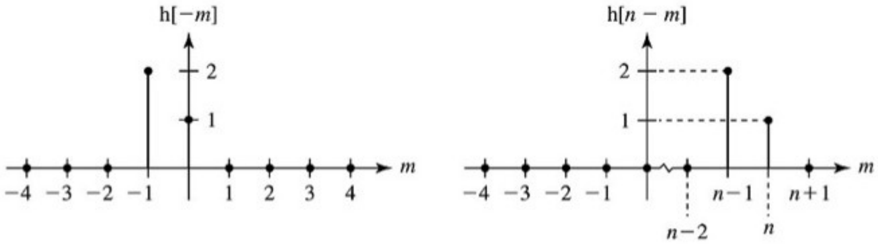



**Figure 3.22**

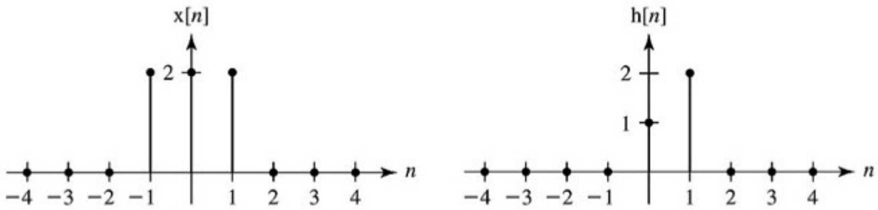



**Figure 3.23:** Two DT signals $x[n]$ and $h[n]$.

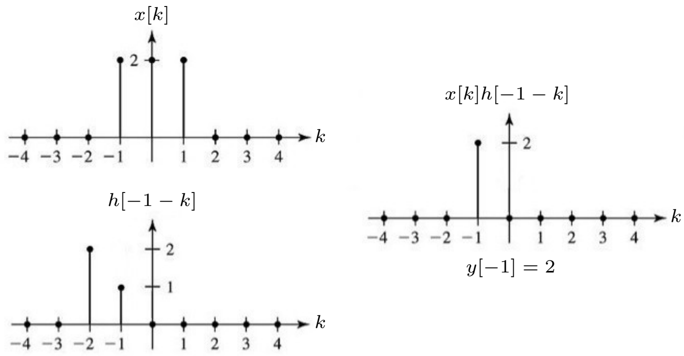
**Example 3.5**

Determine $y[n] = x[n] * h[n]$ graphically, where $x[n]$ and $h[n]$ are defined Figure 3.16 of Example 3.4 and reproduced here for convenience.

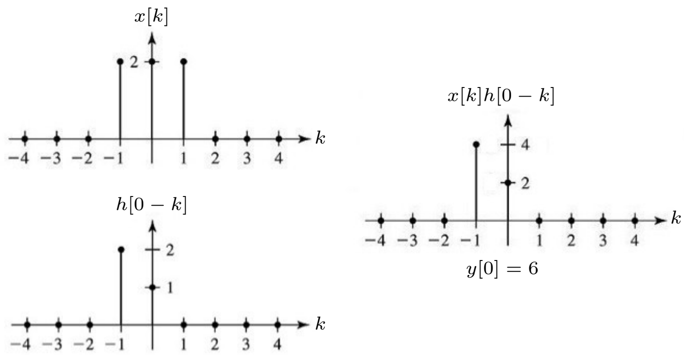
■ **Solution** Before starting with the graphical procedure it is a good idea here to determine where the first sample in the output will appear (this was found earlier to be at $n = -1$). Furthermore, the width property implies that the number of elements in $y[n]$ are four samples. Thus, $y[n] = 0$ for $-\infty < n \leq -2$, and $y[n] = 0$ for $n \geq 3$, hence the only interesting range for $n$ is $-1 \leq n \leq 2$. Now for $n = -1$

$$y[-1] = \sum_{k=-\infty}^{\infty} x[k]h[-1-k]$$

and realizing a negative $n$ ($n = -1$) implies a time shift to the left for the function $h[-1-k]$. Next multiply $x[k]$ by $h[-1-k]$ and add all the products to obtain $y[-1] = 2$ as illustrated in Figure 3.24(a). We keep repeating the procedure incrementing $n$ by one every time, it is important to note here that by incrementing $n$ by one every time means shifting $h[n-k]$ to the right by one sample. Figures 3.24(b), (c) and (d) illustrate the procedure for $n = 0, 1$ and 2 respectively. We can now graph $y[n]$ as illustrated in Figure 3.25. ■

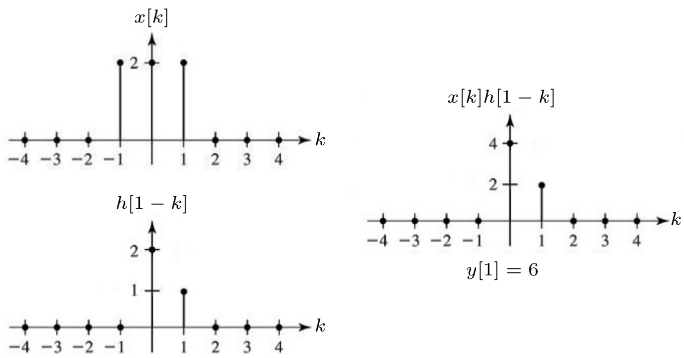



(a) $y[n]$ for $n = -1$.

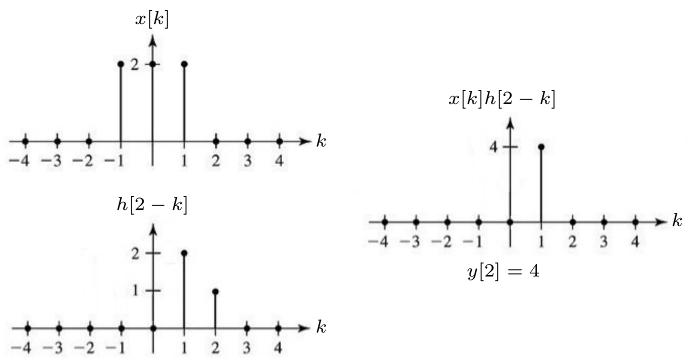



(b) $y[n]$ for $n = 0$.

**Figure 3.24:**
$y[n]$ for $n = -1, 0, 1$, and 2.




(c) $y[n]$ for $n = 1$.




(d) $y[n]$ for $n = 2$.

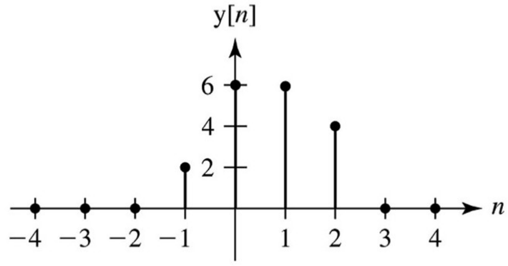



**Figure 3.25:** Graph of $y[n]$.

### Alternative Form of Graphical Procedure

The alternative procedure is basically the same, the only difference is that instead of presenting the data as graphical plots, we display it as a sequence of numbers in a table form. The following example demonstrates the idea and should become clearer.

**Example 3.6**

Determine $y[n] = x[n] * h[n]$, using a tabular form method where $x[n]$ and $h[n]$ are defined as in Figure 3.16.

In this procedure since both sequences $x[n]$ and $h[n]$ are finite, we can perform the convolution easily by setting up a table of values $x[k]$ and $h[n-k]$ for the relevant values of $n$, and using

$$y[n] = \sum_{-\infty}^{\infty} x[k]h[n-k]$$

as shown in Table 3.1. The entries for $h[n-k]$ in the table are obtained by first reflecting $h[k]$ about the origin ($n = 0$) to form $h[-k]$. Before going any further we have to align the rows such that the first element in the stationary $x[k]$ row corresponds to the first element of the already inverted $h[-k]$ row as illustrated in Table 3.1 We now successively shift the inverted row by 1 slot to the right. $y[n]$ is determined by multiplying the entries in the rows corresponding to $x[k]$ and $h[n-k]$ and summing the results. Thus, to find $y[-1]$ multiply the entries in rows 1 and 3; for $y[0]$, multiply rows 1 and 4; and so on. ■

**Table 3.1:** Convolution Table for Example 3.6

| $k$ | -2 | -1 | 0 | 1 | 2 | $n$ | $y[n]$ |
|---|---|---|---|---|---|---|---|
| $x[k]$ | | 2 | 2 | 2 | | | |
| $h[k]$ | | | 1 | 2 | | | |
| $h[-1-k]$ | 2 | 1 | | | | -1 | 2 |
| $h[-k]$ | | 2 | 1 | | | 0 | 6 |
| $h[1-k]$ | | | 2 | 1 | | 1 | 6 |
| $h[2-k]$ | | | | 2 | 1 | 2 | 4 |

## 3.4 The Convolution Integral

Let us turn our attention now to CT LTI systems, we shall use the principle of superposition to derive the system's response to some arbitrary input $x(t)$. In this approach, we express $x(t)$ in terms of impulses. Suppose the CT signal $x(t)$ in Figure 3.26 is an arbitrary input to some CT system. We begin by

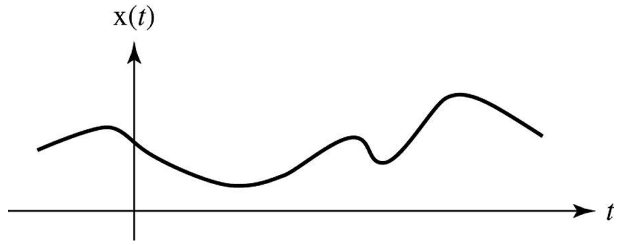



**Figure 3.26:** An arbitrary input.

approximating $x(t)$ with narrow rectangular pulses as depicted in Figure 3.27. This procedure gives us a staircase approximation of $x(t)$ that improves as pulse width is reduced. In the limit as pulse width approaches zero, this representation

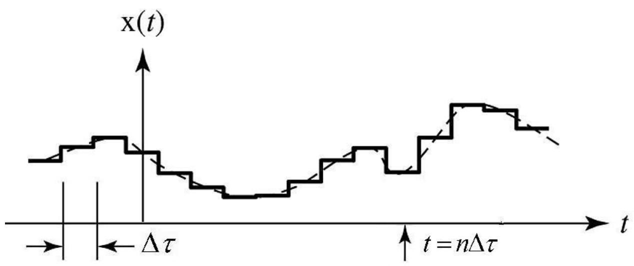



**Figure 3.27:** Staircase approximation to an arbitrary input.

becomes exact, and the rectangular pulses becomes impulses delayed by various amounts. The system response to the input $x(t)$ is then given by the sum of the system's responses to each impulse component of $x(t)$. Figure 3.27 shows $x(t)$ as a sum of rectangular pulses, each of width $\Delta\tau$. In the limit as $\Delta\tau \to 0$, each pulse approaches an impulse having a strength equal to the area under that pulse. For example the pulse located at $t = n\Delta\tau$ can be expressed as

$$x(n\Delta\tau)\,\mathrm{rect}\left(\frac{t - n\Delta\tau}{\Delta\tau}\right)$$

and will approach an impulse at the same location with strength $x(n\Delta\tau)\Delta\tau$, which can be represented by

$$\underbrace{[x(n\Delta\tau)\Delta\tau]}_{\text{strength}}\,\delta(t - n\Delta\tau) \tag{3.14}$$

If we know the impulse response of the system $h(t)$, the response to the impulse in (3.14) will simply be $[x(n\Delta\tau)\Delta\tau]h(t-n\Delta\tau)$ since

$$\begin{aligned} \delta(t) &\xrightarrow{\mathcal{H}} h(t) \\ \delta(t-n\Delta\tau) &\xrightarrow{\mathcal{H}} h(t-n\Delta\tau) \\ [x(n\Delta\tau)\Delta\tau]\delta(t-n\Delta\tau) &\xrightarrow{\mathcal{H}} [x(n\Delta\tau)\Delta\tau]h(t-n\Delta\tau) \end{aligned} \tag{3.15}$$

The response in (3.15) represents the response to only one of the impulse components of $x(t)$. The total response $y(t)$ is obtained by summing all such components (with $\Delta\tau \to 0$)

$$\underbrace{\lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty} x(n\Delta\tau)\Delta\tau\delta(t-n\Delta\tau)}_{\text{The input } x(t)} \xrightarrow{\mathcal{H}} \underbrace{\lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty} x(n\Delta\tau)\Delta\tau h(t-n\Delta\tau)}_{\text{The output } y(t)}$$

and both sides by definition are integrals given by

$$\underbrace{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}_{x(t)} \xrightarrow{\mathcal{H}} \underbrace{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}_{y(t)}$$

In summary the response $y(t)$ to the input $x(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{3.16}$$

and the integral in (3.16) is known as the *convolution integral* and denoted by $y(t) = x(t) * h(t)$.

### Properties of The Convolution Integral

The properties of the convolution integral are the same as of the convolution sum and will be stated here for completion.

1. The Commutative Property

$$x(t) * h(t) = h(t) * x(t) \tag{3.17}$$

This property can be easily proven by starting with the definition of convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

and letting $\lambda = t-\tau$ so that $\tau = t-\lambda$ and $d\tau = -\lambda$, we obtain

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)d\lambda = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = h(t) * x(t)$$

2. The Distributive Property

$$x(t) * (h(t) + z(t)) = x(t) * h(t) + x(t) * z(t) \tag{3.18}$$

3. The Associative Property

$$x(t) * (h(t) * z(t)) = (x(t) * h(t)) * z(t) \tag{3.19}$$

4. The Shifting property

$$x(t - T_1) * h(t - T_2) = y(t - T_1 - T_2) \tag{3.20}$$

5. Convolution with an Impulse

$$x(t) * \delta(t) = x(t) \tag{3.21}$$

   By definition of convolution

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau \tag{3.22}$$

   Because $\delta(t-\tau)$ is an impulse located at $\tau = t$ and according to the sifting property of the impulses, the integral in (3.22) is the value of $x(\tau)$ at $\tau = t$, that is $x(t)$.

6. The Width Property

   If $x(t)$ has a duration of $T_1$ and $h(t)$ has a duration of $T_2$, then the duration of $y(t)$ is $T_1 + T_2$. Furthermore, the appearance of the output will be located at the summation of the times of where the two functions first appear.

7. The Scaling Property

$$\text{If} \quad y(t) = x(t) * h(t) \quad \text{then} \quad y(at) = |a|x(at) * h(at)$$

   This property of the convolution integral has no counterpart for the convolution sum.

### The Graphical procedure

The steps in evaluating the convolution integral are parallel to thoses followed in evaluating the convolution sum. If $y(t)$ is the convolution of $x(t)$ and $h(t)$, then

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \tag{3.23}$$

One of the crucial points to remember here is that the integration is performed with respect to $\tau$, so that $t$ is just like a constant. This consideration is also important when we sketch the graphical representations of the functions $x(t)$ and $h(t - \tau)$. Both of these functions should be sketched as functions of $\tau$, not of $t$. The convolution operation can be performed as follows:

1. Keep the function $x(\tau)$ fixed.
2. Reflect $h(\tau)$ about the vertical axis ($t = 0$) to obtain $h(-\tau)$.
3. Time shift $h(-\tau)$ by $t_0$ seconds to obtain $h(t_0 - \tau)$. For $t_0 > 0$, the shift is to the right; for $t_0 < 0$, the shift is to the left.

4. Find the area under the product of $x(\tau)$ by $h(t_0 - \tau)$ to obtain $y(t_0)$, the value of the convolution at $t = t_0$.

5. The procedure is repeated for each value of $t$ over the range $-\infty$ to $\infty$.

**Example 3.7** Determine $y(t) = x(t) * h(t)$ for $x(t)$ and $h(t)$ as shown in Figure 3.28.

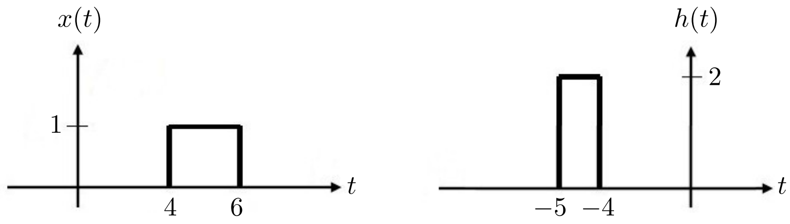



**Figure 3.28:** CT signals to be convolved

■ **Solution**

- (Step 1) Replace $t$ with $\tau$ in $x(t)$ and $h(t)$.
- (Step 2) Choose to flip $h(\tau)$ to obtain $h(-\tau)$ while keeping $x(\tau)$ fixed. Figure 3.29(a) shows $x(\tau)$ and $h(-\tau)$ as functions of $\tau$. The function $h(t - \tau)$ (Figure 3.29(b)) is now obtained by shifting $h(-\tau)$ by $t$. If $t$ is positive, the shift is to the right; if $t$ is negative the shift is to the left. Recall, that convolution is commutative, therefore, we could fix $h(\tau)$ and flip $x(\tau)$ instead.

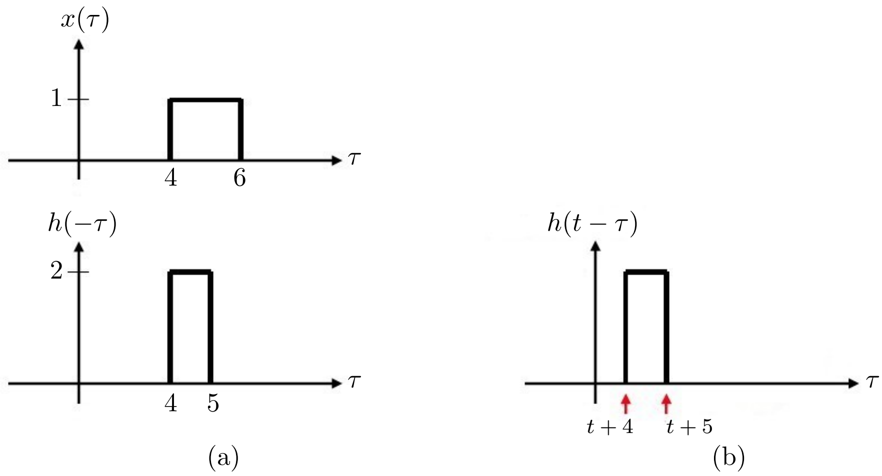



**Figure 3.29:** CT signals to be convolved

- (Step 3) Identify the regions of interest to evaluate the convolution integral, taking particular care to the limits of the convolution integral.

  I. (Region I) Figure 3.29 shows that for $t + 5 < 4 \implies t < -1$, $h(t - \tau)$ does not overlap $x(\tau)$, and the product $x(\tau)h(t - \tau) = 0$, so that $y(t) = 0$ for $t < -1$.

II. (Region 2) The region of interest here could be defined as follows:

$$\begin{aligned} t+5 \geq 4 &\implies t \geq -1 \\ t+4 < 4 &\implies t < 0 \\ &\implies -1 \leq t < 0 \end{aligned}$$

as clearly illustrated in Figure 3.30. Here, $x(\tau)$ and $h(t-\tau)$ do overlap and the product is nonzero only over the interval $4 \leq \tau \leq t+5$ (shaded area). Next, we find the area under the product of the two functions (Figure 3.30). Therefore,

$$\begin{aligned} y(t) &= \underbrace{\int_{4}^{t+5} 2\, d\tau}_{\text{Area}} = 2\tau \Big|_{4}^{t+5} = 2[(t+5)-4] \\ &= 2t+2, \quad -1 \leq t < 0 \end{aligned}$$

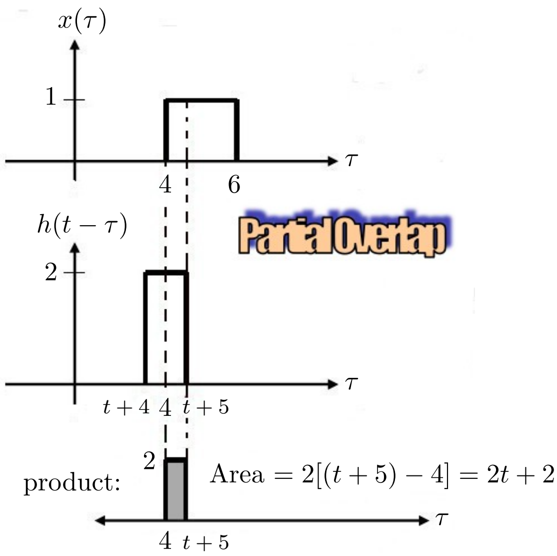



**Figure 3.30:** Convolution of $x(t)$ and $h(t)$.

- (STEP 4) We keep right shifting $h(-\tau)$ to obtain $h(t-\tau)$ to cover all regions of interest. The next interesting range is:

III. (Region 3) defined as follows:

$$\begin{aligned} t+4 \geq 4 &\implies t \geq 0 \\ t+5 < 6 &\implies t < 1 \\ &\implies 0 \leq t < 1 \end{aligned}$$

as clearly illustrated in Figure 3.32. Here, $x(\tau)$ and $h(t-\tau)$ do overlap and the product is nonzero only over the interval $t+4 \leq \tau \leq t+5$

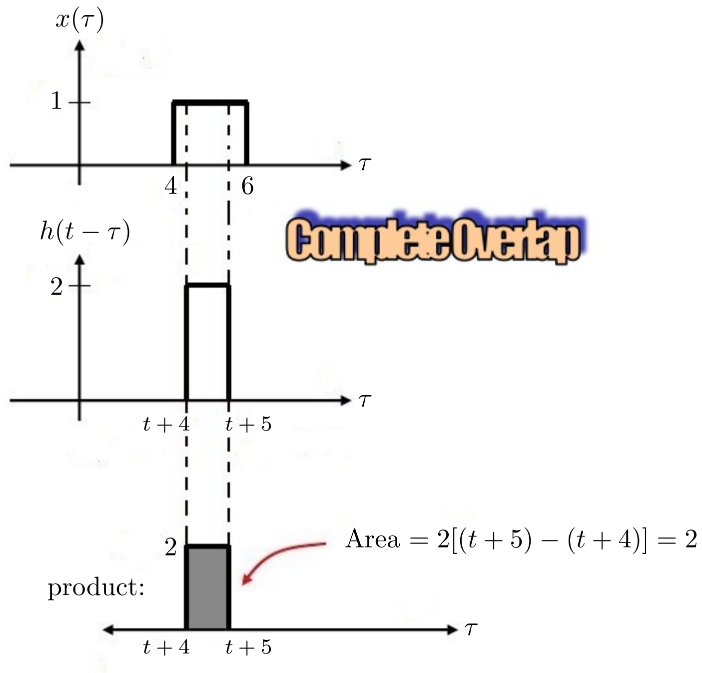



**Figure 3.31:** Convolution of $x(t)$ and $h(t)$.

(shaded area). Next, we find the area under the product of the two functions (Figure 3.32). Therefore,

$$y(t) = \underbrace{\int_{t+4}^{t+5} 2\,d\tau}_{\text{Area}} = 2\tau\Big|_{t+4}^{t+5} = 2[(t+5)-(t+4)]$$
$$= 2, \quad 0 \le t < 1$$

IV. (Region 4): As we keep right shifting $h(-\tau)$, the next region of interest is:

$$t+5 \ge 6 \implies t \ge 1$$
$$t+4 < 6 \implies t < 2$$
$$\implies 1 \le t < 2$$

as clearly illustrated in Figure 3.32. Here, $x(\tau)$ and $h(t-\tau)$ do overlap and the product is nonzero only over the interval $t+4 \le \tau \le 6$ (shaded area). Next, we find the area under the product of the two functions (Figure 3.32). Therefore,

$$y(t) = \underbrace{\int_{t+4}^{6} 2\,d\tau}_{\text{Area}} = 2\tau\Big|_{t+4}^{6} = 2[(6)-(t+4)]$$
$$= 4-2t, \quad 1 \le t < 2$$

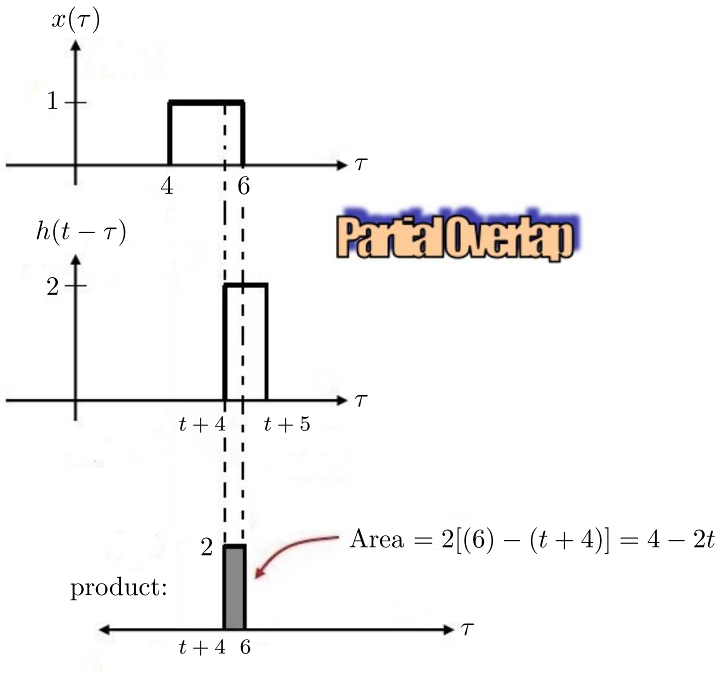



**Figure 3.32:** Convolution of $x(t)$ and $h(t)$.

V. (Region 5) It is clear for $t+4 \geq 6 \implies t \geq 2$, $x(\tau)$ will not overlap $h(t-\tau)$ which implies $y(t) = 0$ for $t \geq 2$.

- (STEP 5) We assemble all the regions together, therefore the result of the convolution is (Figure 3.33),

$$y(t) = \begin{cases} 0, & t < -1 \\ 2t+2, & -1 \leq t < 0 \\ 2, & 0 \leq t < 1 \\ 4-2t, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \quad \blacksquare$$

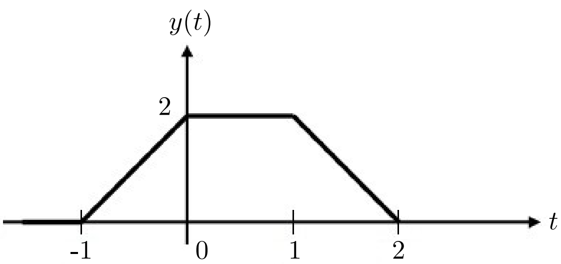



**Figure 3.33:** Convolution of $x(t)$ and $h(t)$.

**Remark** To check your answer, the convolution has the property that the area under the convolution integral is equal to the product of the areas of the two signals entering into the convolution. The area can be computed by integrating equation (3.16) over the interval $-\infty < t < \infty$, giving

$$\begin{aligned}\int_{-\infty}^{\infty} y(t)dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau dt \\ &= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)dt\right] d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\,[\text{area under } h(t)]\, d\tau \\ &= \text{area under } x(t) \times \text{area under } h(t)\end{aligned}$$

This check also applies to DT convolution,

$$\sum_{n=-\infty}^{\infty} y[n] = \sum_{m=-\infty}^{\infty} x[m] \sum_{n=-\infty}^{\infty} h[n-m]$$

**Example 3.8**

For an LTI system with impulse response $h(t) = e^{-t}u(t)$, determine graphically the response $y(t)$ for the input

$$x(t) = \text{rect}\left(\frac{t+1.5}{3}\right)$$

■ **Solution**

- (STEP 1) Replace $t$ with $\tau$ in $x(t)$ and $h(t)$.
- (STEP 2) Choose to flip $x(\tau)$ to obtain $x(-\tau)$ while keeping $h(\tau)$ fixed, since $x(\tau)$ is simpler and symmetric. Figure 3.34(a) shows $h(\tau)$ and $x(-\tau)$ as functions of $\tau$. The function $x(t-\tau)$ (Figure 3.34(b)) is now obtained by shifting $x(-\tau)$ by $t$.

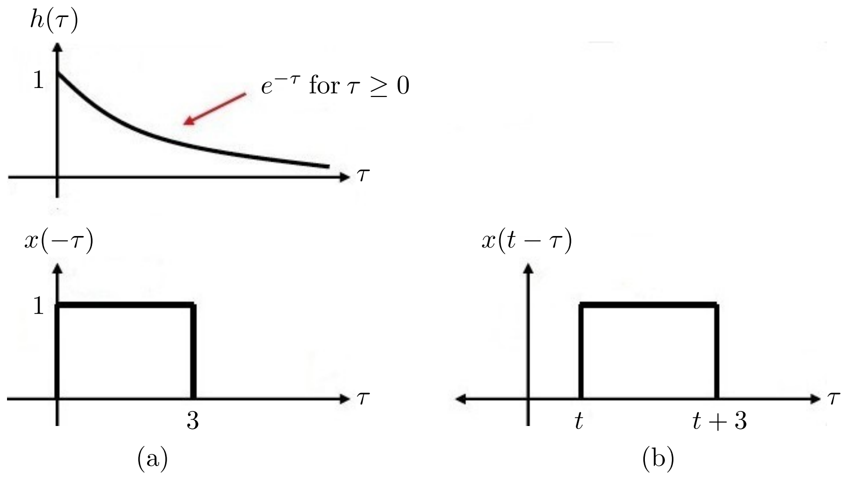



**Figure 3.34:** CT signals to be convolved

- (STEP 3) Identify the regions of interest to evaluate the convolution integral, taking particular care to the limits of the convolution integral.

I. (Region I) Figure 3.34 shows that for $t+3<0 \implies t<-3$, $x(t-\tau)$ does not overlap $h(\tau)$, and the product $h(\tau)x(t-\tau)=0$, so that $y(t)=0$ for $t<-3$.

II. (Region 2) The region of interest here could be defined as follows:

$$\begin{aligned} t+3 \geq 0 &\implies t \geq -3 \\ t<0 &\implies t<0 \\ &\implies -3 \leq t < 0 \end{aligned}$$

as clearly illustrated in Figure 3.35. Here, $h(\tau)$ and $x(t-\tau)$ do overlap and the product is nonzero only over the interval $0 \leq \tau \leq t+3$ (shaded area). Next, we find the area under the product of the two functions (Figure 3.35). Therefore,

$$\begin{aligned} y(t) &= \underbrace{\int_0^{t+3} e^{-\tau} d\tau}_{\text{Area}} = -e^{-\tau}\Big|_0^{t+3} = -[e^{-(t+3)}-1] \\ &= 1-e^{-(t+3)}, \quad -3 \leq t < 0 \end{aligned}$$

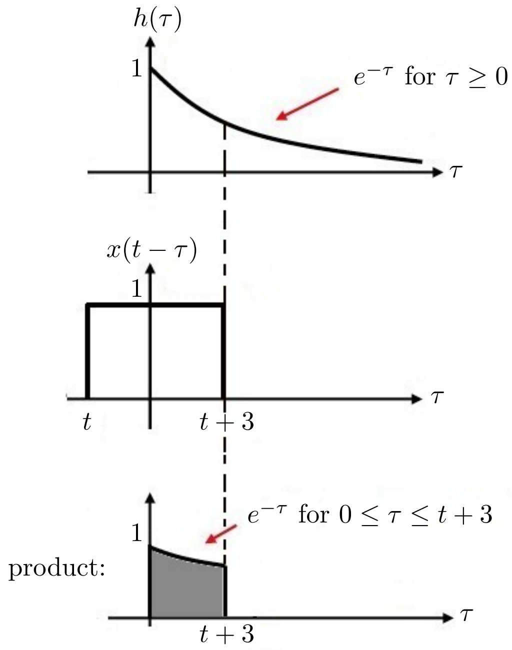



**Figure 3.35:** Convolution of $x(t)$ and $h(t)$.

- (Step 4) We keep right shifting $x(-\tau)$ to obtain $x(t-\tau)$ to cover all regions of interest. The next and final interesting range is:

III. (Region 3) defined as follows: $t \geq 0$ as illustrated in Figure 3.36. Here, $x(\tau)$ and $h(t-\tau)$ do overlap and the product is nonzero only

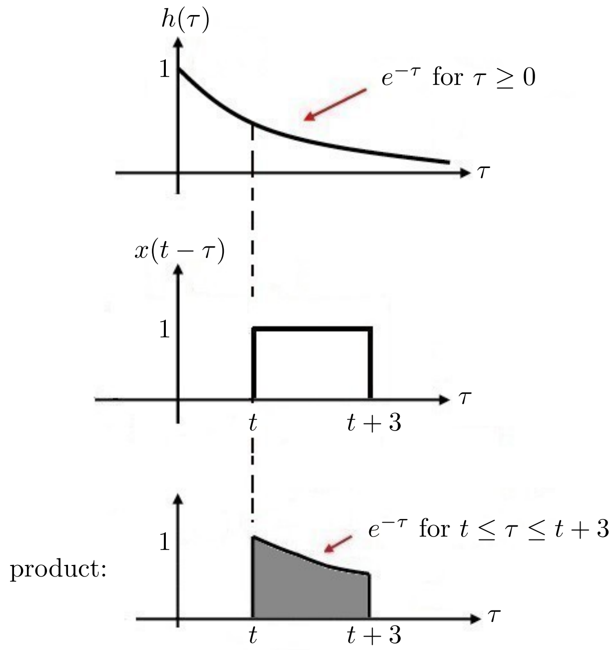



**Figure 3.36:** Convolution of $x(t)$ and $h(t)$.

over the interval $t \leq \tau \leq t+3$ (shaded area). Next, we find the area under the product of the two functions (Figure 3.36). Therefore,

$$y(t) = \underbrace{\int_{t}^{t+3} e^{-\tau} d\tau}_{\text{Area}} = -e^{-\tau}\Big|_{t}^{t+3} = -[e^{-(t+3)} - e^{-t}]$$

$$= e^{-t} - e^{-3}e^{-t}, \quad t \geq 0$$

- (Step 5) We assemble all the regions together, therefore the result of the convolution is (Figure 3.37),

$$y(t) = \begin{cases} 0, & t < -3 \\ 1 - e^{-(t+3)}, & -3 \leq t < 0 \\ (1 - e^{-3})e^{-t}, & t \geq 0 \end{cases} \qquad \blacksquare$$

## 3.5 Properties of LTI Systems

In Section 3.2, several properties of CT systems are defined. In this section, we investigate these properties as related to the impulse response. The impulse response of an LTI system represents a complete description of the characteristics of the system. Hence, all properties of a system can be determined from $h(t)$.

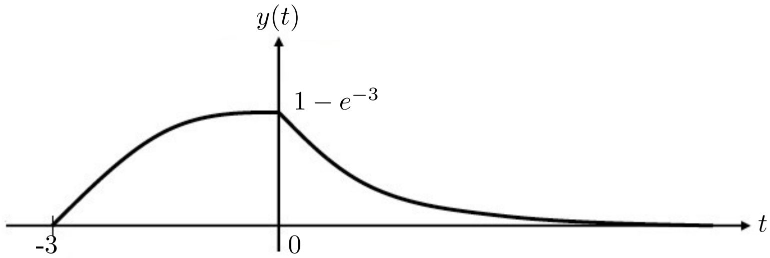



**Figure 3.37:** Convolution of $x(t)$ and $h(t)$.

### Memoryless LTI Systems

In section 3.2.1 we defined a system to be memoryless if its output at any instant in time depends only on the values of the input at the same instant in time. There we saw the input output relation of a memoryless LTI system is

$$y(t) = Kx(t) \tag{3.24}$$

for some constant $K$. By setting $x(t) = \delta(t)$ in (3.24), we see that this system has the impulse response

$$h(t) = K\delta(t)$$

IHence, an LI system is memoryless if and only if $h(t) = K\delta(t)$. Memoryless systems are what we call *constant gain* systems.

### Invertible LTI Systems

Recall that a system is invertible only if there exists an *inverse* system which enables the reconstruction of the input given the output. If $h_{inv}(t)$ represents the impulse response of the inverse system, then in terms of the convolution integral we must therefore have

$$y(t) = x(t) * h(t) * h_{inv}(t) = x(t)$$

this is only possible if

$$h(t) * h_{inv}(t) = h_{inv}(t) * h(t) = \delta(t)$$

### Causal LTI Systems

A CT LTI system is causal if the current value of the output depends on only the current and past values of the input. Because the unit impulse function $\delta(t)$ occurs at $t = 0$, the impulse response $h(t)$ of a causal system must be zero for $t < 0$. The convolution integral of a LTI system can be expressed as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

for a CT system to be causal, $y(t)$ must not depend on $x(\tau)$ for $\tau > t$. We can see that this will be so if

$$h(t-\tau) = 0 \quad for \quad \tau > t$$

Let $\lambda = t - \tau$, implies

$$h(\lambda) = 0 \quad for \quad \lambda < 0$$

In this case the convolution integral becomes

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \\ &= \int_{0}^{\infty} x(t-\tau)h(\tau)d\tau \end{aligned}$$

Stable LTI Systems

A CT system is stable if and only if every bounded input produces a bounded output. Consider a bounded input $x(t)$ such that $|x(t)| < B$ for all $t$. Suppose that this input is applied to an LTI system with impulse response $h(t)$. Then

$$\begin{aligned} |y(t)| &= \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \\ &\leq \int_{-\infty}^{\infty} |h(\tau)| \, |x(t-\tau)| \, d\tau \\ &\leq B \int_{-\infty}^{\infty} |h(\tau)| \, d\tau \end{aligned}$$

Therefore, because $B$ is finite, $y(t)$ is bounded, hence, the system is stable if

$$\int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty \tag{3.25}$$

**Example 3.9** For an LTI system with impulse response $h(t) = e^{-3t}u(t)$, determine the stability of this causal LTI system.

■ **Solution** For an LTI causal system, the stability criterion in (3.25) reduces to

$$\int_{0}^{\infty} |h(\tau)| \, d\tau < \infty$$

hence

$$\int_{0}^{\infty} e^{-3t}dt = -\frac{1}{3}e^{-3t}\Big|_{0}^{\infty} = \frac{1}{3} < \infty$$

and this system is stable. ■

Chapter 4

# The Fourier Series

In Chapter 3 we saw how to obtain the response of a linear time invariant system to an arbitrary input represented in terms of the impulse function. The response was obtained in the form of the convolution integral. In this chapter we explore other ways of expressing an input signal in terms of other signals. In particular we are interested in representing signals in terms of complex exponentials, or equivalently, in terms of sinusoidal (sine and cosine) waveforms. This representation of signals leads to the Fourier series, named after the French physicist Jean Baptiste Fourier. Fourier was the first to suggest that periodic signals could be represented by a sum of sinusoids. The concept is really simple: consider a periodic signal with fundamental period $T_0$ and fundamental frequency $\omega_0 = 2\pi f_0$, this periodic signal can be expressed as a linear combination of harmonically related sinusoids as shown in Figure 4.1. In the Fourier

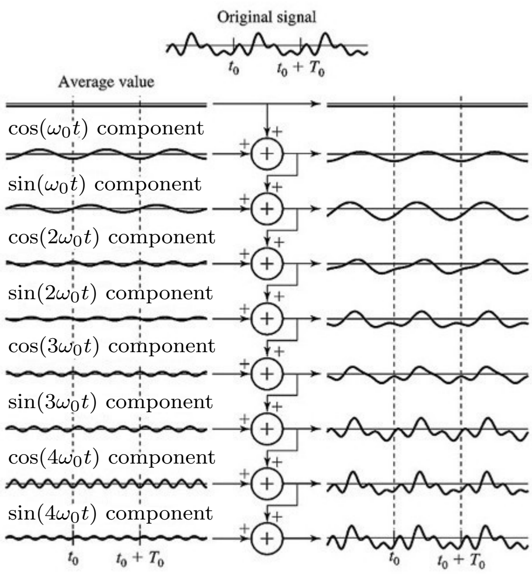



**Figure 4.1:** The concept of representing a periodic signal as a linear combination of sinusoids

series representation of a signal, the higher frequency components of sinusoids have frequencies that are integer multiplies of the fundamental frequency. This is called the *harmonic number*, for example, the function $\cos(k\omega_0 t)$ is the $k^{th}$ harmonic cosine, and its frequency is $k\omega_0$ rad/sec. or $kf$ in Hertz. The idea of the Fourier series demonstrated in Figure 4.1 uses a constant, sines and cosines to represent the original function, thus called the *trigonometric* Fourier series. Another form of the Fourier series is the *complex* form, here the original periodic function is represented as a combination of harmonically related complex exponentials. A set of harmonically related complex exponentials form an orthogonal basis by which periodic signals can be represented, a concept explored next.

## 4.1 Orthogonal Representations of Signals

In this section we show a way of representing a signal as a sum of orthogonal signals, such representation simplifies calculations involving signals. We can visualize the signal as a vector in an orthogonal coordinate system, with the orthogonal waveforms being the unit coordinates. Let us begin with some basic vector concepts and then apply these concepts to signals.

### 4.1.1 Orthogonal Vector Space

Vectors, functions and matrices can be expressed in more than one set of coordinates, which we usually call a vector space. A three dimensional Cartesian is an example of a vector space denoted by $\mathcal{R}^3$, as illustrated in Figure 4.2. The three axes of three-dimensional Cartesian coordinates, conventionally denoted the $x$, $y$, and $z$ axes form the *basis* of the vector space $\mathcal{R}^3$. A very natural and simple basis is simply the vectors $\mathbf{x}_1 = (1, 0, 0)$, $\mathbf{x}_2 = (0, 1, 0)$ and $\mathbf{x}_3 = (0, 0, 1)$. Any vector in $\mathcal{R}^3$ can be written in terms of this basis. A vector $\mathbf{v} = (a, b, c)$ in $\mathcal{R}^3$ for example can be written uniquely as the linear combination $\mathbf{v} = a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3$ and $a, b, c$ are simply called the *coefficients*. We can obtain the coefficients with respect to the basis using the inner product, for vectors this is simply the dot product

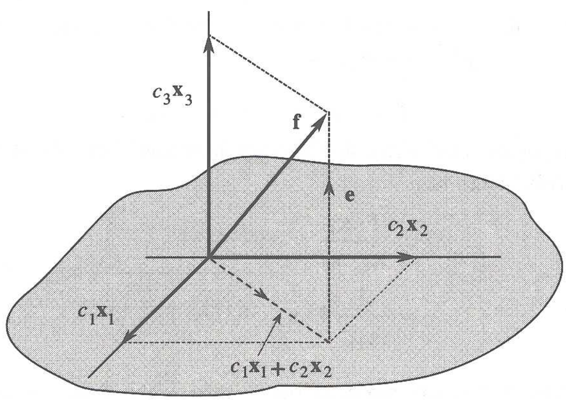



**Figure 4.2:** Representation of a vector in three-dimensional space.

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