There are many probability distributions that have been studied over the years. Each of which was developed based on observations of a particular physical phenomenon. The table below shows some of these distributions:

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<td>Maxwell-Boltzmann</td>
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<td>Phase-type</td>
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<td>type-1 Gumbel</td>
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<td>Type-2 Gumbel</td>
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<td>Gompertz</td>
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<td>Uniform</td>
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</tr>
</tbody>
</table>

In this article we will summarize the characteristics of some popular probability distributions, which you are likely to encounter later in your study of communication engineering. We will begin with discrete distributions then move to continuous distributions.

A - Discrete Random Variables:

- **Bernoulli distribution**

  The Bernoulli random variable takes value 1 with success probability $p$ and value 0 with failure probability $q = 1 - p$.

  **Applications:**
  - Coin toss.
  - Success/Failure experiments.
• Binomial distribution

The binomial random variable is the sum of \( n \) independent, identically distributed Bernoulli random variables, each of which yields success with probability \( p \). In fact, when \( n = 1 \), the binomial random variable is a Bernoulli random variable.

Applications:
• Obtaining \( k \) heads in \( n \) tossings of a coin.
• Receiving \( k \) bits correctly in \( n \) transmitted bits.
• Batch arrivals of \( k \) packets from \( n \) inputs at an ATM switch.
**Probability mass function**

**Cumulative distribution function**

![Graphs of probability mass and cumulative distribution functions.](image)

**Parameters**

- \( n \geq 0 \) number of trials (integer)
- \( 0 \leq p \leq 1 \) success probability (real)

**Support**

- \( k \in \{0, \ldots, n\} \)

**Probability mass function (pmf)**

\[
\binom{n}{k} p^k (1-p)^{n-k}
\]

**Cumulative distribution function (CDF)**

\[
\sum_{j=0}^{\lfloor x \rfloor} \binom{n}{j} p^j (1-p)^{n-j}
\]

**Mean**

\( np \)

**Median**

one of \( \lfloor np \rfloor, \lfloor np \rfloor + 1 \)

**Mode**

\( \lfloor (n+1)p \rfloor \)

**Variance**

\( np(1-p) \)

**Skewness**

\[
\frac{1 - 2p}{\sqrt{np(1-p)}}
\]

**Excess kurtosis**

\[
\frac{1 - 6p(1-p)}{np(1-p)}
\]

**Probability-generating function (PGF)**

\( (1 + pz)^n \)

**Moment-generating function (MGF)**

\( (1 - p + pe^t)^n \)

**Characteristic function**

\( (1 - p + pe^{it})^n \)

---

**Geometric distribution**

This is the distribution of time slots between the successes of consecutive *independent* Bernoulli trails. Like its continuous analogue (the exponential distribution), the geometric distribution is *memoryless*, which is an important property in certain applications.

**Applications:**

- To represent the number of dice rolls needed until you roll a six.
- Queueing theory and discrete Markov chains.
### Parameters

Support

Probability mass function \((\text{pmf})\)

Cumulative distribution function \((\text{CDF})\)

Mean

Median

Mode

Variance

Skewness

Excess kurtosis

Probability-generating function \((\text{PGF})\)

Moment-generating function \((\text{MGF})\)

Characteristic function

\[
0 < p \leq 1 \quad \text{success probability (real)}
\]

\[k \in \{1, 2, 3, \ldots\}\]

\[ (1 - p)^{k-1} p \]

\[ 1 - (1 - p)^{\lfloor k \rfloor} \]

\[ \frac{1}{p} \]

\[ \left\lfloor -\frac{\ln(2)}{\ln(1 - p)} \right\rfloor \]

\[ 1 \]

\[ \frac{1 - p}{p^2} \]

\[ \frac{2 - p}{\sqrt{1 - p}} \]

\[ 6 + \frac{p^2}{1 - p} \]

\[ \frac{pz}{1 - (1 - p)z} \]

\[ \frac{pe^t}{1 - (1 - p)e^t} \]

\[ pe^{it} \]

\[ 1 - (1 - p)e^{it} \]

---

**Poisson distribution**

The **Poisson distribution** describes the probability that a number \(k\) of events occur in a fixed period of time assuming these events occur at *random* with a rate \(\lambda\).

**Applications:**
- The number of phone calls at a call center per minute.
- The number of times a web server is accessed per minute.
• The number of spelling mistakes one makes while typing a single page.
• The number of mutations in a given stretch of DNA after radiation.
• The number of unstable nuclei that decayed within a given period of time in a piece of radioactive substance.
• The number of stars in a given volume of space.

![Probability mass function](image1)
![Cumulative distribution function](image2)

**Parameters**

- $\lambda \in (0, \infty)$ rate (real)
- $k \in \{0, 1, 2, \ldots\}$
- $e^{-\lambda} \frac{\lambda^k}{k!}$
- $\frac{\Gamma([k + 1], \lambda)}{[k]!}$ for $k \geq 0$ where $\Gamma(x, y)$ is the Incomplete gamma function

**Mean**

- $\lambda$

**Median**

- usually about $[\lambda + 1/3 - 0.02/\lambda]$

**Mode**

- $\lfloor \lambda \rfloor$ (and $\lambda - 1$ if $\lambda$ is an integer)

**Variance**

- $\lambda$

**Skewness**

- $\lambda^{-1/2}$

**Excess kurtosis**

- $\lambda^{-1}$

**Probability-generating function (PGF)**

- $e^{\lambda(x-1)}$

**Moment-generating function (MGF)**

- $e^{\lambda(e^t-1)}$

**Characteristic function**

- $e^{\lambda(e^{it}-1)}$

• **Zipf distribution**

The term Zipf's law refers to frequency distributions of rank data. Originally, Zipf's law stated that, in natural language utterances, the frequency of any word is roughly inversely proportional to its rank in the frequency table. So, the most frequent word will occur approximately twice as often as the second most frequent word, which occurs twice as often as the fourth most frequent word, etc.
Applications:

- Similar to that of Poisson distribution (e.g., the time it takes before your next telephone call, the distance between mutations on a DNA strand, etc).

B - Continuous Random Variables:

- **Exponential distribution**

  The exponential distribution represents the probability distribution of the time intervals between successive Poisson arrivals. The exponential distribution is the continuous counterpart of the geometric distribution, and is the only continuous distribution that is memoryless.

  Applications:

  - Similar to that of Poisson distribution (e.g., the time it takes before your next telephone call, the distance between mutations on a DNA strand, etc).
• Uniform distribution

The uniform distribution is often abbreviated \( U(a, b) \). In this distribution, all time intervals of the same length are equally probable.

Applications:
• To test a statistic for the simple null hypothesis.
• In simulation software packages, in which uniformly-distributed pseudo-random numbers are generated first and then converted to other types of distributions.
Parameters  \( a, b \in (-\infty, \infty) \)  lower/upper limits

Support  \( a \leq x \leq b \)

\[
\frac{1}{b-a} \quad \text{for } a \leq x \leq b
\]

Probability density function \( (pdf) \)

\[
0 \quad \text{for } x < a \text{ or } x > b
\]

\[
\frac{x-a}{b-a} \quad \text{for } a \leq x < b
\]

\[
1 \quad \text{for } x \geq b
\]

Cumulative distribution function \( (CDF) \)

Mean  \( \frac{a + b}{2} \)

Median  \( \frac{a + b}{2} \)

Mode  any value in \( [a, b] \)

Variance  \( \frac{(b-a)^2}{12} \)

Skewness  \( 0 \)

Excess kurtosis  \( -\frac{6}{5} \)

Moment-generating function \( (MGF) \)

\[
\frac{e^{tb} - e^{ta}}{t(b-a)} \quad \text{The raw moments are: } \quad m_k = \frac{1}{k+1} \sum_{i=0}^{k} a^i b^{k-i}.
\]

Characteristic function

---

**Triangular distribution**

The Triangular distribution is used as a subjective description of a population for which there is only limited sample data.

**Applications:**
- Used in business decision making and project management to describe the time to completion of a task.

---

![Probability density function](image1)

![Cumulative distribution function](image2)

Parameters  
- \( a : a \in (-\infty, \infty) \) lower limit
- \( b : b > a \) upper limit
- \( c : a \leq c \leq b \) mode
Support

\[ a \leq x \leq b \]

Probability density function (pdf)

\[
\begin{align*}
&\frac{2(x-a)}{(b-a)(c-a)} \text{ for } a \leq x \leq c \\
&\frac{2(b-x)}{(b-a)(b-c)} \text{ for } c \leq x \leq b \\
\end{align*}
\]

Cumulative distribution function (CDF)

\[
\begin{align*}
&\frac{(x-a)^2}{(b-a)(c-a)} \text{ for } a \leq x \leq c \\
&1 - \frac{(b-x)^2}{(b-a)(b-c)} \text{ for } c \leq x \leq b \\
\end{align*}
\]

Mean

\[
a + b + c \over 3
\]

Median

\[
\begin{align*}
&a + \frac{\sqrt{(b-a)(c-a)}}{\sqrt{2}} \text{ for } c \geq \frac{b-a}{2} \\
&b - \frac{\sqrt{(b-a)(b-c)}}{\sqrt{2}} \text{ for } c \leq \frac{b-a}{2}
\end{align*}
\]

Mode

\[ c \]

Variance

\[
\frac{a^2 + b^2 + c^2 - ab - ac - bc}{18}
\]

Skewness

\[
\frac{\sqrt{2}(a+b-2c)(2a-b-c)(a-2b+c)}{5(a^2+b^2+c^2-ab-ac-bc)^{3/2}}
\]

Excess kurtosis

\[
\frac{3}{5}
\]

Moment-generating function (MGF)

\[
\frac{2(b-c)e^{st} - (b-a)e^{at} + (c-a)e^{bt}}{(b-a)(c-a)(b-c)t^2}
\]

Characteristic function

\[
-2\frac{(b-c)e^{i\theta} - (b-a)e^{ia} + (c-a)e^{ib}}{(b-a)(c-a)(b-c)t^2}
\]

- **Normal distribution**

The **normal distribution** is also called the **Gaussian distribution**, and is denoted by \( N(\mu, \sigma^2) \). The **standard normal distribution** is the normal distribution with a **mean** of zero and a **variance** of one. It is often called the **bell curve** because the graph of its probability density resembles a bell.

**Applications:**
- Many physical phenomena (like **noise**) can be approximated well by the normal distribution.
- Many statistical tests are based on the assumption of normal distribution.
- Normal distribution arises as the limiting distribution of several continuous and discrete families of distributions by the **central limit theorem**.
- Measurement errors are often assumed to be normally distributed.
- Financial variables such as stock values or commodity prices can be modeled by the normal distribution.
- Light intensity from a single source is usually assumed to be normally distributed.
The green line is the standard normal distribution

Parameters
- $\mu$ location / mean (real)
- $\sigma^2 > 0$ squared scale / variance (real)

Support $x \in \mathbb{R}$

Probability density function (pdf)
\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \]

Cumulative distribution function (CDF)
\[ F(x) = \frac{1}{2} \left( 1 + \text{erf}\left(\frac{x-\mu}{\sigma \sqrt{2}}\right) \right) \]

Mean $\mu$

Median $\mu$

Mode $\mu$

Variance $\sigma^2$

Skewness 0

Excess kurtosis 0

Moment-generating function (MGF)
\[ M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \]

Characteristic function
\[ \phi(t) = \exp\left(\mu it - \frac{\sigma^2 t^2}{2}\right) \]

- **Log-normal distribution**

If $Y$ is a random variable with a normal distribution, then $X = \exp(Y)$ has a log-normal distribution; likewise, if $X$ is log-normally distributed, then $\ln(X)$ is normally distributed.

**Applications:**
- The long-term return rate on a stock investment can be modeled as log-normal.
Parameters

\(-\infty \leq \mu \leq \infty\)  mean

\(\sigma \geq 0\)  standard deviation

Support

\(x \in [0; +\infty)\)

Probability density function (pdf)

\[ f(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp \left( -\frac{[\ln(x) - \mu]^2}{2\sigma^2} \right) \]

Cumulative distribution function (CDF)

\[ F(x; \mu, \sigma) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\ln(x) - \mu}{\sigma\sqrt{2}} \right) \]

Mean

\(e^{\mu + \sigma^2/2}\)

Median

\(e^\mu\)

Mode

\(e^{\mu - \sigma^2}\)

Variance

\((e^{\sigma^2} - 1)e^{2\mu + \sigma^2}\)

Skewness

\((e^{\sigma^2} + 2)\sqrt{e^{\sigma^2} - 1}\)

Excess kurtosis

\[ \frac{e^{2\sigma^2} - 4e^{\sigma^2} + 6e^{\sigma^2} - 3}{e^{4\mu + 2\sigma^2}(e^{\sigma^2} - 1)^2} \]

Moment-generating function (MGF)

The MGF does not exist, but moments do, and are given by:

\[ m_k = e^{k\mu + k^2\sigma^2/2} \]

\- Gamma (Erlang) distribution

A gamma distributed random variable is denoted by \(\Gamma(k, \lambda)\). The Erlang distribution is a special case of the gamma distribution where the shape parameter \(k\) is an integer. In the Gamma distribution, this parameter is a real number.

Applications:

\- The number of telephone calls which might be made at the same time to a switching center.
Applications:

- Used in common chi-square tests for goodness of fit of observed data to a theoretical distribution.
- Used in statistical significance tests. One example is Friedman's analysis of variance by ranks.

**chi-square distribution**

The chi-square distribution ($\chi^2$ distribution) is a special case of the gamma distribution where $k = n/2$ and $\lambda = 1/2$. 

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$k &gt; 0$</th>
<th>shape (real)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda &gt; 0$</td>
<td>rate (real)</td>
<td></td>
</tr>
</tbody>
</table>

**Support**

$x \in [0; \infty)$

**Probability density function (pdf)**

$$f(x; k, \lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$$

where the gamma function $\Gamma(k) = \int_0^\infty y^{k-1} e^{-y} dy$

becomes $\Gamma(k) = (k - 1)!$ if $k$ is an integer.

**Cumulative distribution function (CDF)**

$$F(x; k, \lambda) = \frac{\gamma(x; k, \lambda)}{\Gamma(k)}$$

**Mean**

$$k/\lambda$$

**Median**

no simple closed form

**Mode**

$$(k - 1)/\lambda \text{ for } k \geq 1$$

**Variance**

$$k/\lambda^2$$

**Skewness**

$$\frac{2}{\sqrt{k}}$$

**Excess kurtosis**

$$\frac{6}{k}$$

**Moment-generating function (MGF)**

$$(1 - t/\lambda)^{-k} \text{ for } t < \lambda$$

**Characteristic function**

$$(1 - it/\lambda)^{-k}$$
Rayleigh distribution

The Rayleigh distribution has been used to model attenuation of wireless signals facing multi-path fading. The Chi, Rice and Weibull distributions are all generalizations of the Rayleigh distribution.
Parameters

Support

Probability density function (pdf)

Cumulative distribution function (CDF)

Mean

Median

Mode

Variance

Skewness

Excess kurtosis

Moment-generating function (MGF)

Characteristic function

**Student’s t distribution**

The Student’s t distribution arises in the problem of estimating the mean of a normally distributed population when the sample size is small.

**Applications:**

- It is the basis of the popular Student's t-tests for the statistical significance of the difference between two sample means.
Probability density function (pdf) \[ f(x) = \frac{\Gamma((\nu + 1)/2)}{\sqrt{\pi} \Gamma(\nu/2)} \left(1 + x^2/\nu\right)^{-(\nu+1)/2} \]

Cumulative distribution function (CDF) \[ F(x) = \frac{1}{2} + \frac{\nu}{\nu - 2} \frac{2F_1 \left(\frac{1}{2}, (\nu + 1)/2; \frac{3}{2}; -x^2/\nu\right)}{\sqrt{\pi \nu \Gamma(\nu/2)}} \] where \(2F_1\) is the hypergeometric function.

Mean \(0\) for \(\nu > 1\), otherwise undefined
Median \(0\)
Mode \(0\)
Variance \(\frac{\nu}{\nu - 2} \begin{cases} \frac{4\pi^2}{3} & \text{for } \nu > 2, \\ \text{undefined} & \text{otherwise} \end{cases}\)
Skewness \(0\) for \(\nu > 3\)
Excess kurtosis \(\frac{6}{\nu - 4}\) for \(\nu > 4\)

**Cauchy distribution**

**Applications:**
- In spectroscopy the Cauchy distribution is the description of the line shape of spectral lines which are broadened by collisions.
- In nuclear and particle physics, the energy profile of a resonance is described by the relativistic Cauchy distribution.
• **Pareto distribution**

The Pareto distribution is characterized by a curved *long tail* when plotted on a linear scale, thus is it used to describe long-range dependent phenomena.

**Applications:**
- Used to describe the allocation of wealth among individuals.
- Describes frequencies of words in longer texts (a few words are used often, lots of words are used infrequently).
- File size distribution of Internet traffic which uses the TCP protocol (many smaller files, few larger ones).
- Clusters of Bose-Einstein condensate near absolute zero.
- The values of oil reserves in oil fields (a few large fields, many small fields)
- The length distribution in jobs assigned to supercomputers (a few large ones, many small ones).
- The standardized price returns on individual stocks.
- Areas burnt in forest fires.

![Probability density function](image1)

![Cumulative distribution function](image2)

**Parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location (real)</td>
<td>$x &gt; 0$</td>
</tr>
<tr>
<td>Shape (real)</td>
<td>$k &gt; 0$</td>
</tr>
</tbody>
</table>

**Support**

$x \in [x_m, +\infty)$

**Probability density function (pdf)**

$$f(x) = \frac{k x_m^k}{x^{k+1}}$$

**Cumulative distribution function (CDF)**

$$F(x) = 1 - \left(\frac{x_m}{x}\right)^k$$

**Mean**

$$\frac{k x_m}{k - 1} \text{ for } k > 1$$

**Median**

$$x_m \sqrt[3]{2}$$

**Mode**

$$x_m$$

**Variance**

$$\frac{x_m^2 k}{(k - 1)^2(k - 2)} \text{ for } k > 2$$
Skewness

\[ \frac{2(1 + k)}{k - 3} \sqrt{\frac{k - 2}{k}} \]  
for \( k > 3 \)

Excess kurtosis

\[ \frac{6(k^3 + k^2 - 6k - 2)}{k(k - 3)(k - 4)} \]  
for \( k > 4 \)

• **Laplace distribution**

![Probability density function](image.png)  
![Cumulative distribution function](image.png)

**Parameters**

- \( \mu \): location (real)
- \( b > 0 \): scale (real)

**Support**

\( x \in (-\infty; +\infty) \)

**Probability density function (pdf)**

\[
\frac{1}{2b} \exp \left( - \frac{|x - \mu|}{b} \right)
\]

\[
= \begin{cases} 
\frac{1}{2b} \exp \left( - \frac{x - \mu}{b} \right) & \text{if } x < \mu \\
\frac{1}{2b} \exp \left( - \frac{\mu - x}{b} \right) & \text{if } x \geq \mu 
\end{cases}
\]

**Cumulative distribution function (CDF)**

\[
\begin{cases} 
\frac{1}{2} \exp \left( - \frac{x - \mu}{b} \right) & \text{if } x < \mu \\
1 - \frac{1}{2} \exp \left( - \frac{\mu - x}{b} \right) & \text{if } x \geq \mu 
\end{cases}
\]

- **Mean**: \( \mu \)
- **Median**: \( \mu \)
- **Mode**: \( \mu \)
- **Variance**: \( 2b^2 \)
- **Skewness**: 0
- **Excess kurtosis**: 3

**Moment-generating function (MGF)**

\[
\frac{\exp(\mu t)}{1 - b^2 t^2} \quad \text{for } |t| < 1/b
\]

**Characteristic function**

\[
\frac{\exp(\mu i t)}{1 + b^2 t^2}
\]
Notes:
The expected value for a discrete random variable is:

$$\mu = E[X] = \sum_{i} p_i x_i$$

For a continuous random variable:

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x) \, dx$$

The **variance** of a random variable is a measure of its statistical dispersion, indicating how its possible values are spread around the expected value. For a random variable $X$, the variance is:

$$\text{Var}(X) = E((X - \mu)^2).$$

If the random variable is discrete the variance is equivalent to:

$$\text{Var}[X] = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$$

The **mode** is the most frequent value assumed by a random variable, or occurring in a sampling of a random variable. The mode is not necessarily unique, since the same maximum frequency may be attained at different values.

The **median** is the number separating the higher half of a sample or a probability distribution, from the lower half. The median of a finite list of numbers can be found by arranging all the observations from lowest value to highest value and picking the middle one. If there is an even number of observations, the median is not unique.

**Skewness** is a measure of the asymmetry of the probability density function.

**Kurtosis** (from the Greek word kurtos) is a measure of the peakedness of the probability density function.