Process Heat Transfer

Lec 2: Heat Conduction Equation

Fourier's Law and the Heat Equation:
Developing the general unsteady-state equation for the different coordinates

Dr. Zayed Al-Hamamre

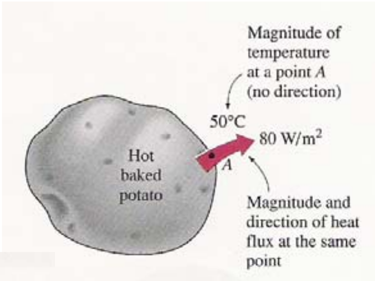
Content

- Multidimensionality and time dependence of heat transfer, and the conditions under which a heat transfer problem can be approximated as being one-dimensional,
- The differential equation of heat conduction in various coordinate systems, and simplify it for steady one-dimensional case,
- The thermal conditions on surfaces, and express them mathematically as boundary and initial conditions,
- Solve one-dimensional heat conduction problems and obtain the temperature distributions within a medium and the heat flux,
- Analyze one-dimensional heat conduction in solids that involve heat generation, and
- Evaluate heat conduction in solids with temperature-dependent thermal conductivity.

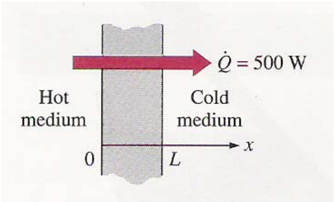
Introduction

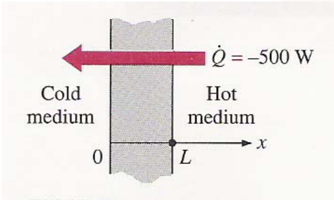
- Heat transfer has *direction* as well as *magnitude*

  ⟹ ***q* is a vector**

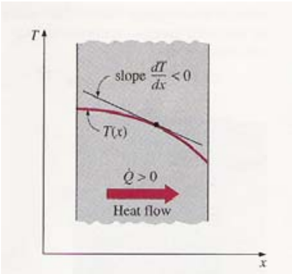


- Sign convention for heat transfer:





- Conduction is proportional to the *temperature gradient*.

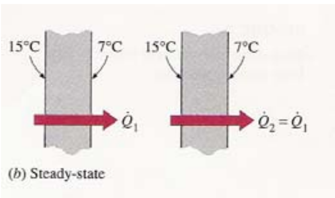


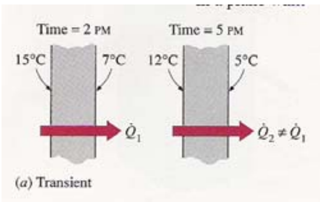
Introduction

Classification of conduction problems: in general $T = T(x, y, z, t)$

⇒ Steady state versus transient conduction;

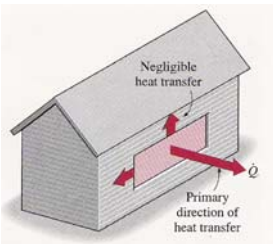
$T = T(x)$





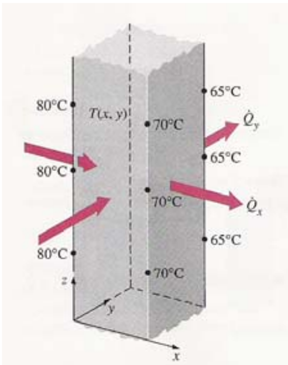
$T = T(x, t)$

⇒ One-dimensional versus two dimensional conduction



$T = T(x)$

$T = T(x, y)$



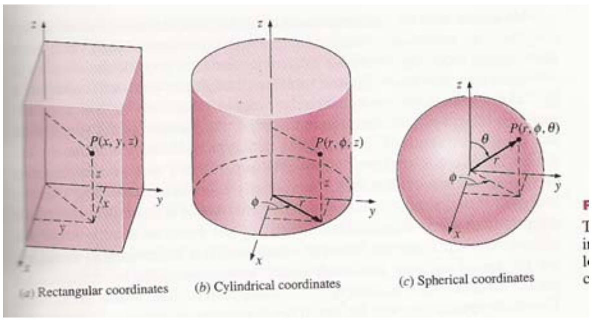
Introduction

⇒ Three-dimensional conduction

⇒ Rectangular: $T = T(x, y, z)$

Cylindrical: $T = T(r, \theta, z)$

Cylindrical: $T = T(r, \theta, \varphi)$



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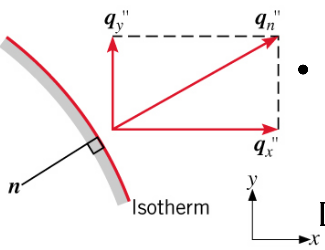
Fourier's Law

- A rate equation that allows determination of the conduction heat flux from knowledge of the temperature distribution in a medium
- Its most general (vector) form for multidimensional conduction is:

Implications:

- Heat transfer is in the direction of decreasing temperature (basis for minus sign).



- Fourier's Law serves to define the thermal conductivity of the medium

Direction of heat transfer is perpendicular to lines of constant temperature (isotherms).

- Heat flux vector may be resolved into orthogonal components.

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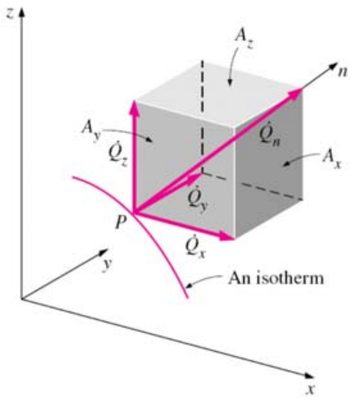
General Relation for Fourier's Law of Heat Conduction

- In rectangular coordinates, the heat conduction vector can be expressed in terms of its components as

$$\vec{\dot{Q}}_n = \dot{Q}_x\vec{i} + \dot{Q}_y\vec{j} + \dot{Q}_z\vec{k}$$

- which can be determined from Fourier's law as

$$\begin{cases} \dot{Q}_x = -kA_x \dfrac{\partial T}{\partial x} \\ \dot{Q}_y = -kA_y \dfrac{\partial T}{\partial y} \\ \dot{Q}_z = -kA_z \dfrac{\partial T}{\partial z} \end{cases}$$



Heat Generation

- Examples:
  - electrical energy being converted to heat at a rate of I^2R,
  - fuel elements of nuclear reactors,
  - exothermic chemical reactions.
- Heat generation is a *volumetric phenomenon*.
- The rate of heat generation units : W/m^3 or $Btu/h \cdot ft^3$.
- The rate of heat generation in a medium may vary with time as well as position within the medium.
- The *total* rate of heat generation in a medium of volume V can be determined from

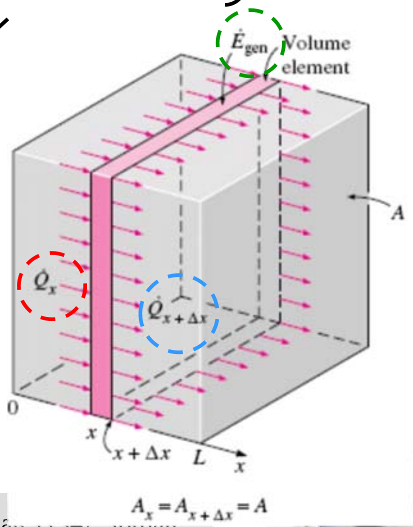
$$\dot{E}_{gen} = \int_V \dot{e}_{gen}\, dV \qquad \text{(W)}$$

One-Dimensional Heat Conduction Equation - Plane Wall

Applying conservation of energy to a differential control volume through which energy transfer is exclusively by conduction.

(Rate of heat conduction at x) - (Rate of heat conduction at $x+\Delta x$) + (Rate of heat generation inside the element) = (Rate of change of the energy content of the element)

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$



One-Dimensional Heat Conduction Equation - Plane Wall

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

- The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \Delta E_{element} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \rho c A \Delta x \left(T_{t+\Delta t} - T_t\right) \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta x \end{cases}$$

- Substituting into Eq. 2–6, we get

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{e}_{gen} A \Delta x = \rho c A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

One-Dimensional Heat Conduction Equation - Plane Wall

Dividing by $A\,\Delta x$, taking the limit as $\Delta x \rightarrow 0$ and $\Delta t \rightarrow 0$,

and from Fourier's law $\dot{Q}_x = -kA\dfrac{\partial T}{\partial x}$

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

A differential equation whose solution provides the temperature distribution in a stationary medium.

- The area A is constant for a plane wall → the one dimensional transient heat conduction equation in a plane wall is

Variable conductivity:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}\quad;\quad \alpha=\frac{k}{\rho c}$$

, Thermal diffusivity of the medium

One-Dimensional Heat Conduction Equation - Plane Wall

- The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state:

$$\frac{d^2T}{dx^2}+\frac{\dot{e}_{gen}}{k}=0$$

2) Transient, no heat generation:

$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

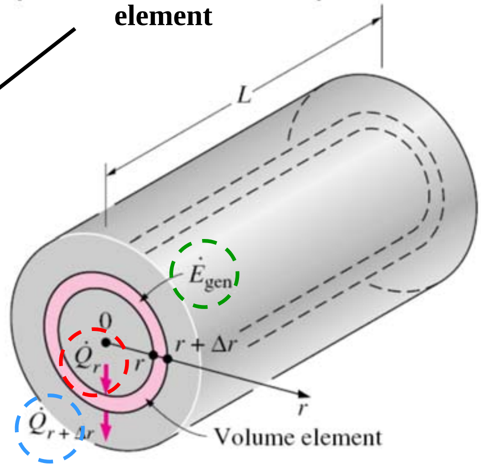
3) Steady-state, no heat generation:

$$\frac{d^2T}{dx^2}=0$$

One-Dimensional Heat Conduction Equation - Long Cylinder

(Rate of heat conduction at r) - (Rate of heat conduction at $r+\Delta r$) + (Rate of heat generation inside the element) = (Rate of change of the energy content of the element)

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$



- The change in the energy content and the rate of heat generation can be expressed as

$$\begin{cases} \Delta E_{element} = E_{t+\Delta t} - E_t = mc\left(T_{t+\Delta t} - T_t\right) = \rho c A \Delta r \left(T_{t+\Delta t} - T_t\right) \\ \dot{E}_{gen,element} = \dot{e}_{gen} V_{element} = \dot{e}_{gen} A \Delta r \end{cases}$$

One-Dimensional Heat Conduction Equation - Long Cylinder

- Substituting into Eq. 2–18, we get

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{e}_{gen} A \Delta r = \rho c A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t}$$

- Dividing by $A\,\Delta r$, taking the limit as $\Delta r \rightarrow 0$ and $\Delta t \rightarrow 0$,

  and from Fourier's law:

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t}$$

➢ Noting that the area varies with the independent variable r according to $A = 2\pi r L$, the one dimensional transient heat conduction equation in a long cylinder becomes

One-Dimensional Heat Conduction Equation - Long Cylinder

Variable conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

- The one-dimensional conduction equation may be reduces to the following forms under special conditions

1) Steady-state: $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{gen}}{k}=0$$

2) Transient, no heat generation: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3) Steady-state, no heat generation: $$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

One-Dimensional Heat Conduction Equation - Sphere

Same approach used for the cylinder with

A = 4 π r^2 instead of A = 2 π rL

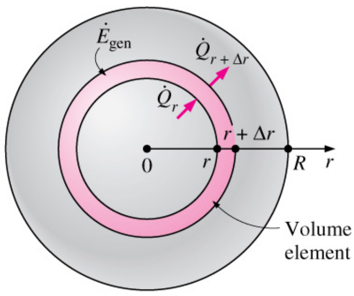


FIGURE 2–17
One-dimensional heat conduction through a volume element in a sphere.

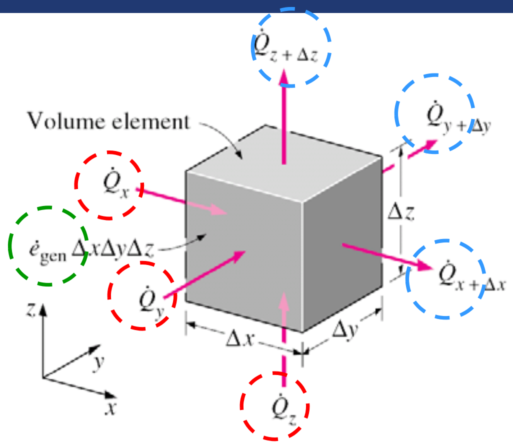
Variable conductivity:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2k\frac{\partial T}{\partial r}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

Constant conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{e}_{gen}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

General Heat Conduction Equation

Cartesian Coordinates:



Rate of heat conduction at *x*, *y*, and *z* − **Rate of heat conduction at *x+Δx*, *y+Δy*, and *z+Δz*** + **Rate of heat generation inside the element** = **Rate of change of the energy content of the element**

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + E_{gen,element} = \frac{\Delta E_{element}}{\Delta t}$$

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General Heat Conduction Equation

➢ Repeating the mathematical approach used for the one-dimensional heat conduction the three-dimensional heat conduction equation is determined to be

Two-dimensional

Constant conductivity:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Three-dimensional

1) Steady-state: **(Poisson equation)**

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{gen}}{k} = 0$$

2) Transient, no heat generation: **(Diffusion equation)**

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

3) Steady-state, no heat generation: **(Laplace equation)**

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

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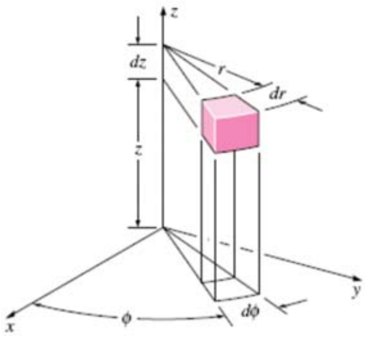
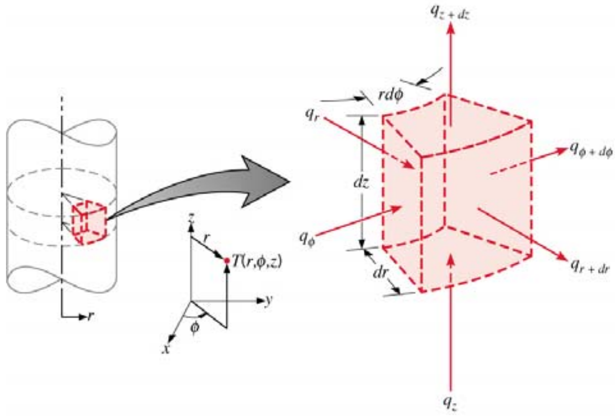
Cylindrical Coordinates

- Cylindrical Coordinates:





$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

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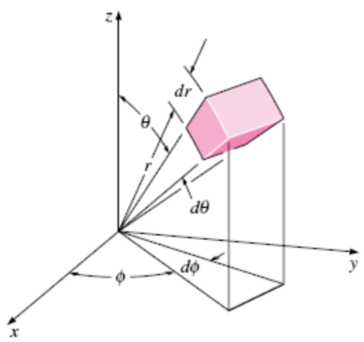
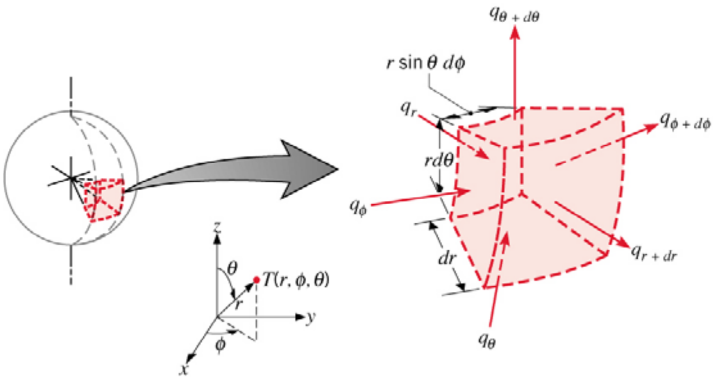
Spherical Coordinates

- Spherical Coordinates:





$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{e}_{gen}=\rho c\frac{\partial T}{\partial t}$$

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Example

The temperature distribution across a wall 1 m thick at a certain instant of time is given as

$$T(x) = a + bx + cx^2$$

where T is in degrees Celsius and x is in meters, while $a = 900°\text{C}$, $b = -300°\text{C/m}$, and $c = -50°\text{C/m}^2$. A uniform heat generation, $\dot{q} = 1000\ \text{W/m}^3$, is present in the wall of area $10\ \text{m}^2$ having the properties $\rho = 1600\ \text{kg/m}^3$, $k = 40\ \text{W/m} \cdot \text{K}$, and $c_p = 4\ \text{kJ/kg} \cdot \text{K}$.

1. Determine the rate of heat transfer entering the wall ($x = 0$) and leaving the wall ($x = 1$ m).
2. Determine the rate of change of energy storage in the wall.
3. Determine the time rate of temperature change at $x = 0$, 0.25, and 0.5 m.

Example cont.

Example cont.

Example cont.

Example cont.

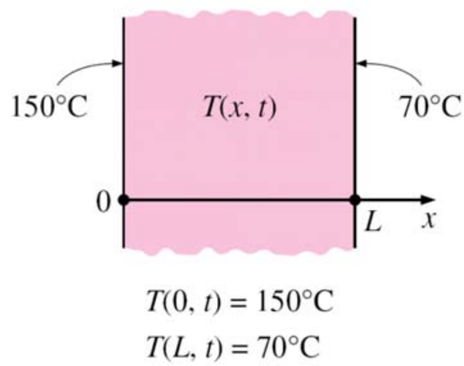
Example cont.

Boundary and Initial Conditions

- Specified Temperature Boundary Condition
- Specified Heat Flux Boundary Condition
- Convection Boundary Condition
- Radiation Boundary Condition
- Interface Boundary Conditions
- Generalized Boundary Conditions

Specified Temperature Boundary Condition

- For one-dimensional heat transfer through a plane wall of thickness *L*, for example, the specified temperature boundary conditions can be expressed as



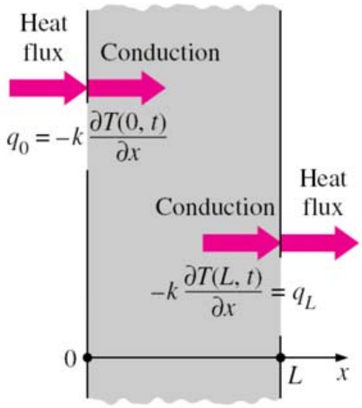
$$T(0, t) = T_1$$

$$T(L, t) = T_2$$

- The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.

Specified Heat Flux Boundary Condition

- The heat flux in the positive *x*-direction anywhere in the medium, including the boundaries, can be expressed by *Fourier's law* of heat conduction as



$$\dot{q} = -k\frac{dT}{dx} = \left(\text{Heat flux in the positive x-direction} \right)$$

- The sign of the specified heat flux is determined by inspection: *positive* if the heat flux is in the positive direction of the coordinate axis, and *negative* if it is in the opposite direction.

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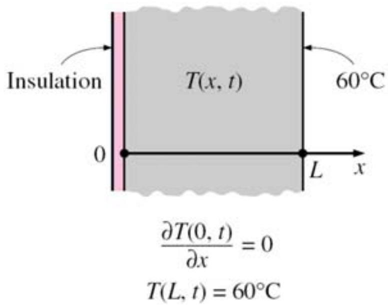
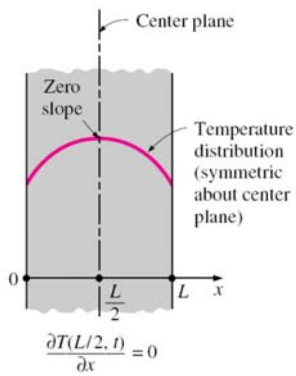
Two Special Cases

Insulated boundary



$$k\frac{\partial T(0,t)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial T(0,t)}{\partial x} = 0$$

Thermal symmetry



$$\frac{\partial T\left(L/2, t\right)}{\partial x} = 0$$

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