$$e^{-\frac{iJ_y\beta}{\hbar}} = \sum_n \frac{1}{n!}\left(-\frac{iJ_y\beta}{\hbar}\right)^n$$

$$= 1 + \left(-\frac{iJ_y\beta}{\hbar}\right) + \frac{1}{2!}(-i\beta)^2\left(\frac{J_y}{\hbar}\right)^2 + \frac{1}{3!}(-i\beta)^3\left(\frac{J_y}{\hbar}\right)^3$$

$$+ \frac{1}{4!}(-i\beta)^4\left(\frac{J_y}{\hbar}\right)^4 + \frac{1}{5!}(-i\beta)^5\left(\frac{J_y}{\hbar}\right)^5 + \frac{1}{6!}(-i\beta)^6\left(\frac{J_y}{\hbar}\right)^6$$

$$+ \frac{i}{7!}(-i\beta)^7\left(\frac{J_y}{\hbar}\right)^7 + \frac{1}{8!}(-i\beta)^8\left(\frac{J_y}{\hbar}\right)^8 + \cdots$$

$$J_y = \frac{J_+ - J_-}{2!} = -\frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\Rightarrow \frac{J_y}{\hbar} = \frac{-i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\left(\frac{J_y}{\hbar}\right)^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\left(\frac{J_y}{\hbar}\right)^3 = \frac{-i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{J_y}{\hbar}$$

$$\left(\frac{J_y}{\hbar}\right)^4 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \left(\frac{J_y}{\hbar}\right)^2$$

$$\left(\frac{J_y}{\hbar}\right)^5 = \frac{-i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{J_y}{\hbar}$$

$$\left(\frac{J_y}{\hbar}\right)^6 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \left(\frac{J_y}{\hbar}\right)^2$$

$$\left(\frac{J_y}{\hbar}\right)^7 = \frac{-i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

clearly:

$$\left(\frac{J_y}{\hbar}\right)^n = \left(\frac{J_y}{\hbar}\right) \quad \text{for} \quad n = 3, 5, 7, \cdots$$

$$= \left(\frac{J_y}{\hbar}\right)^2 \quad \text{for} \quad n = 2, 4, 6, \cdots$$

$$= 1 \quad \text{for} \quad n = 0$$

$$e^{-i\left(\frac{J_y}{\hbar}\right)\beta} = 1 - i\beta\left(\frac{J_y}{\hbar}\right) - \frac{1}{2!}\beta^2\left(\frac{J_y}{\hbar}\right)^2 + \frac{1}{3!}i\beta^3\left(\frac{J_y}{\hbar}\right)$$

$$+ \frac{1}{4!}\beta^4\left(\frac{J_y}{\hbar}\right)^4 - \frac{i\beta^5}{5!}\left(\frac{J_y}{\hbar}\right) - \frac{\beta^6}{6!}\left(\frac{J_y}{\hbar}\right)^2$$

$$+ \frac{i}{7!}\beta^7\left(\frac{J_y}{\hbar}\right) + \frac{1}{8!}\beta^8\left(\frac{J_y}{\hbar}\right)^8 + \cdots$$

$$e^{-i\left(\frac{J_y}{\hbar}\right)\beta} = \underline{1} - \frac{1}{2!}\beta^2\left(\frac{J_y}{\hbar}\right)^2 + \frac{1}{4!}\beta^4\left(\frac{J_y}{\hbar}\right)^2 - \frac{1}{6!}\beta^6\left(\frac{J_y}{\hbar}\right)^2 + \frac{1}{8!}\beta^8\left(\frac{J_y}{\hbar}\right)^2 + \cdots$$

$$-i\left[\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \frac{\beta^7}{7!} + \cdots\right]\left(\frac{J_y}{\hbar}\right)$$

$$e^{-i\left(\frac{J_y}{\hbar}\right)\beta} = \underline{1} + \left[\left(\frac{J_y}{\hbar}\right)^2 - \left(\frac{J_y}{\hbar}\right)^2 - \frac{1}{2!}\beta^2\left(\frac{J_y}{\hbar}\right)^2 + \frac{1}{4!}\beta^4\left(\frac{J_y}{\hbar}\right)^2 - \cdots\right]$$

$$-\sin\beta\left(\frac{iJ_y}{\hbar}\right)$$

$$= \underline{1} + \left[-1 + \left(1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \cdots\right)\right]\left(\frac{J_y}{\hbar}\right)^2$$

$$-\left(\frac{iJ_y}{\hbar}\right)\sin\beta$$

$$= \underline{1} + (\cos\beta - 1)\left(\frac{J_y}{\hbar}\right)^2 - \left(\frac{iJ_y}{\hbar}\right)\sin\beta$$

$$e^{-i\left(\frac{J_y}{\hbar}\right)\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}(\cos\beta - 1)\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$-\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}\sin\beta$$

$$e^{-i\frac{J_y \beta}{\hbar}} = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ -\frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$