Chapter 3: Probability

- Probability is a measure of uncertainty [Increase Your Chance to Win]
- Basic rules for finding Probabilities
- The Probability can be used as a measure of reliability for an inference

Basic Concepts of Probability

(1) Experiment: Process of Collecting data

- Experiment of tossing 2 coins
- Experiment of throwing 2 dice

(2) Sample space S

All outcomes of the experiment

* Experiment of tossing a coin

$S = \{H, T\}$ H: Head
T: Tail

* Experiment of tossing 2 coins

$S = \{HH, HT, TH, TT\}$

* Experiment of tossing one die

$S = \{1, 2, 3, 4, 5, 6\}$

* Experiment of tossing 2 ~~coins~~ dice

$$S = \left\{ \begin{array}{l} (1,1), (1,2), \ldots, (1,6) \\ (2,1), (2,2), \ldots, (2,6) \\ \vdots \\ (6,1), (6,2), \ldots, (6,6) \end{array} \right\}$$

(3) Sample Point : The most basic outcome of an experiment

- Experiment of tossing 2 coins
  $\Rightarrow$ We have 4 sample points

  1. HH
  2. HT
  3. TH
  4. TT

- Experiment of throwing one die

  1. Observe a 1.
  2. " " 2.
  3. " " 3.
  4. " " 4.
  5. " " 5.
  6. " " 6.

(4) Event

Any collection of sample points

- Experiment of tossing 2 coins

  E_1: getting 2 heads $= \{HH\}$

  E_2: Getting at least one head
  $= \{HT, TH, HH\}$

  E_3: exactly one head
  $= \{HT, TH\}$

- Experiment of throwing 2 dice

E_1 = Sum of points on both faces = 7

$= \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

E_2 = all points where sum > 7

$= \{(2,6),(3,5),(3,6),(4,4),(4,5),(4,6),$
$(5,3),(5,4),(5,5),(5,6),(6,2),$
$(6,3),(6,4),(6,5),(6,6)\}$

(5) Probability

Relative frequency of the events

(i) $0 \leq$ Probability of any event ≤ 1

(ii) $P(\phi) = 0$ where ϕ is an empty set

(iii) $P(S) = 1$ where S is a sample space

(iv) The sum of the probabilities of all sample points $= 1$

How to assign Event Probability?

- Define experiment
- List the Sample space [all sample points]
- Identify collection of sample points in event
- Sum of sample point probabilities

Example 1: For an experiment of tossing 2 coins, define

E = at least one head

What is the prob. of getting at least one head?

$$P(E) = P(HT) + P(TH) + P(HH)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

$$= \frac{3}{4}$$

Example 2: Experiment of throwing 2 dice

What is the prob. of rolling a 7 on a pair of die?

Sample point	Prob.
(1, 6)	1/36
(2, 5)	1/36
(3, 4)	1/36
(4, 3)	1/36
(5, 2)	1/36
(6, 1)	1/36

$$\text{Sum} = \frac{6}{36} = \frac{1}{6}$$

Example 3: What is the probability of rolling at least 9 on a pair of dice?

E: At least 9 points on both dice

	Sample Points	Prob.
Sum = 9	(3, 6) (4, 5) (5, 4) (6, 3)	$\frac{4}{36}$
Sum = 10	(4, 6) (5, 5) (6, 4)	$\frac{3}{36}$
Sum = 11	(5, 6) (6, 5)	$\frac{2}{36}$
Sum = 12	(6, 6)	$\frac{1}{36}$

$$\frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36}=\frac{10}{36}=\frac{5}{18}=\boxed{0.28}$$

Combinations

What do you do when the # of sample points is too large?

$\Rightarrow$ Counting Process "مبدأ العد"

Let $A = \{a, b, c\}$. Choose 2 elements from the list

$\{a, b\}$ $\{a, c\}$ $\{b, c\}$ $\Rightarrow$ 3 methods

- Here we consider $\{a, b\}$ as $\{b, a\}$ and for this, we list only one of them.

- Let N = # of points in a set

  n = # of points to be chosen from N

# of different choices of choosing n out of

$$\binom{N}{n} = \cancel{N(N-1)\cdots(3)(2)(1)}$$

$$= \frac{N!}{n!\,(N-n)!}$$

Where $n! = n(n-1)\cdots(3)(2)(1)$

- For example, $N = 4$, $n = 2$

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2 \times 1}{(2 \times 1)(2 \times 1)} = \boxed{6}$$

- Quick computation

$$\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5\times 4\times \cancel{3!}}{2!\,\cancel{3!}} = \boxed{10},$$

or

$$\binom{5}{2} = \frac{5\times 4}{1\times 2} = \boxed{10},$$

Also

$$\binom{10}{3} = \frac{10\times \overset{3}{\cancel{9}}\times \overset{4}{\cancel{8}}}{1\times \cancel{2}\times \cancel{3}} = \boxed{120}$$

Example 4: How many ways of choosing a committee of 3 persons from a group of 8 persons?

$$\binom{8}{3} = \frac{8\times 7\times 6}{1\times 2\times 3} = \boxed{56}$$

Example 5: Given 6 different drugs were compared for their effectiveness in preventing vomiting. Medical researchers looked at all possible combinations of the drug as treatment including a single drug, 2-drug, 3-drug, 4-drug 5-drug & 6-drug combinations.

(a) How many 2-drug combinations of 6 drugs are possible?

$$\binom{6}{2} = \frac{6\times 5}{1\times 2} = 15$$

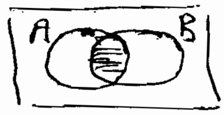
(b) How many 3-drug combinations of 6 drugs are possible?

$$\binom{6}{3} = \frac{6\times 5\times 4}{1\times 2\times 3} = 20$$

Probability Rules

① Union & Intersection

$A \cap B$: all points in A and B



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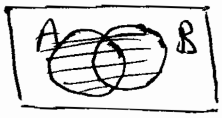
$A \cup B$: " " " A or B



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Example 6: Experiment of tossing a die

A = getting an even #

B = " a number less than or equal to 3.

$\Rightarrow A = \{2, 4, 6\}$

$B = \{1, 2, 3\}$

$A \cap B = \{2\}$ $\qquad$ $A \cup B = \{1, 2, 3, 4, 6\}$

$P(A \cap B) = P(\{2\}) = \frac{1}{6}$

$P(A \cup B) = P(\{1,2,3,4,6\}) = P(\{1\}) + P(\{2\}) + \dots + P(\{6\})$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

$= \frac{5}{6}$

Example 7: Event A : A person has low educational level

Event B : A person is married

Educ. Level

Social life	Low	Middle	High
Married	0.2	0.05	0.05
Single	0.1	0.05	0.15
Divorced	0.15	0.1	0.15

$P(A) = 0.45$

$P(B) = 0.30$

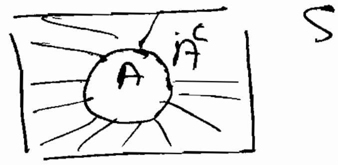
$P(A \cap B) = 0.2$

$P(A \cup B) = P(\text{Married} \cap \text{not Low}) + P(\text{Low} \cap \text{not married}) + P(\text{Married} \cap \text{Low})$

$= 0.10 + 0.25 + 0.25 = 0.55$

(2) Complementary Events

A^c : All points not in A



$$P(A) + P(A^c) = 1$$
$$\Longrightarrow P(A^c) = 1 - P(A)$$

Example 8: Consider the experiment of tossing 2 coins. Let

A: observing at least one head

$A = \{HT, TH, HH\}$

A^c = Observe no heads $= \{TT\}$

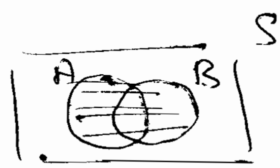
$P(A^c) = P(TT) = \frac{1}{4}$

$P(A) = 1 - P(A^c) = 1 - \frac{1}{4} = \frac{3}{4}$

or $= P(\text{at least one head})$
$= 1 - P(\text{no heads})$
$= 1 - P(TT)$
$= 1 - \frac{1}{4} = 3/4$

(3) Additive Law of Prob.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Look at Example 7

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.45 + 0.30 - 0.2$$
$$= 0.55$$

Def: A & B are mutually exclusive (disjoint) if $A \cap B = \phi$.

Then $P(A \cap B) = 0$

In this case, $\boxed{P(A \cup B) = P(A) + P(B)}$

Example 9: Experiment of tossing 2 coins

A = getting at least one head

B = " exactly one head

C = " " 2 heads

(B, C) disjoint

$A = B \cup C \Rightarrow P(A) = P(B \cup C)$

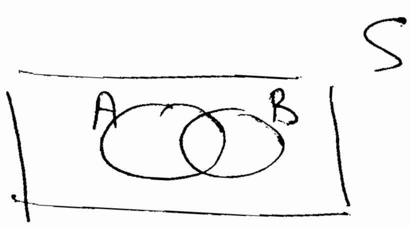
$= P(B) + P(C)$

$= \frac{1}{2} + \frac{1}{4} = 3/4$

(4) Multiplicative Law of Prob.

Conditional Prob.: Prob. that A occurs given B occurs

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$



$$\Rightarrow P(A \cap B) = P(A|B)\,P(B)$$

~~add~~ Multiplicative law of Prob.

Example 10 (Back to Example 9)

Educ. Level

Soc. Life	L	M	H	Tot.
Ma..	0.2	0.05	0.05	0.3
Single	0.1	0.05	0.15	0.3
D..	0.15	0.10	0.15	0.4
Tot.	0.45	0.2	0.35	

A: Educ. level is low

B: Married

$P(A \cap B) = 0.2$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.3} = \frac{2}{3}$$

Example 11:

2 white
1 black

A box contains 2 white balls and one black ball. 2 balls are drawn without replacement.

Find the prob. that the 2 balls are white?

$$P(\text{2 white balls}) = P(\text{1st ball is white and 2nd is white})$$
$$= P(\text{2nd is white} \mid \text{1st is white}) \, P(\text{1st is white})$$
$$= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Note: If A and B are independent, then

$$P(A|B) = P(A) \quad \& \quad P(B|A) = P(B)$$

$$\Rightarrow P(A \cap B) = P(B|A) P(A)$$
$$= P(B) \cdot P(A)$$

Example 12 (Back to Example 9) Are A & B are indep.?

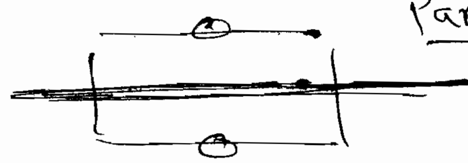
$$P(A|B) = \frac{0.2}{0.3} = \frac{2}{3}$$

$$P(A) = 0.45$$

$\Longrightarrow$ A and B are not indep.

Example 13 : Two components are working independently with $P(A) = 0.3$ & $P(B) = 0.5$. A system of 2 components is working if at least one components is working. What is the prob. that the system is working?

Parallel



$$\begin{aligned} P(\text{Sys. Works}) &= P(\text{at least one component works}) \\ &= P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &= 0.3 + 0.5 - (0.3)(0.5) \\ &= 0.3 + 0.5 - 0.15 \\ &= \boxed{0.65} \end{aligned}$$

or

$$\begin{aligned} P(\text{at least one is working}) &= \cancel{P(A \cup B)} \\ &= 1 - P(\text{no one is working}) \\ &= 1 - P(A^c B^c) \\ &= 1 - (0.7)(0.5) \\ &= 1 - 0.35 \\ &= \boxed{0.65} \end{aligned}$$