Inference about a single population

Confidence Interval

- An estimator $\hat{\theta}$ of θ is unbiased estimator if $E(\hat{\theta}) = \theta$. Otherwise, $\hat{\theta}$ is a biased estimator with bias= $E(\hat{\theta}) - \theta$. The best estimator is the one with minimum variance over the unbiased estimators. Note that S.E.($\hat{\theta}$)= Std. of $\hat{\theta}$.
- $\overline{X}$ is unbiased estimator for μ since $E(\overline{X}) = \mu$.
- Consider the sample mean

$$T_1 = \frac{X_1 + X_2 + X_3}{3}$$

and the weighted average

$$T_2 = \frac{X_1 + 2X_2 + X_3}{4}.$$

Now

$$E(T_1) = \mu, \text{ and } E(T_2) = \mu$$

both T_1 and T_2 are unbiased for μ.

Cont./Confidence Intervals

- $$Var(T_1) = \frac{\sigma^2}{3}, \text{ and } Var(T_2) = \frac{\sigma^2 + 4\sigma^2 + \sigma^2}{16} = \frac{3\sigma^2}{8}$$

  So $S.E.(T_1) < S.E.(T_2)$ and then $\overline{X}$ is a better estimator than T_2.

- (L, U) is called a $100(1-\alpha)\%$ confidence interval (C.I.) for the parameter θ iff

  $$P(L < \theta < U) = 1 - \alpha$$

- Let $X_1, X_2, ..., X_n \sim N(\mu, \sigma^2)$. The sample mean $\overline{X} \sim N(\mu, \sigma^2/n)$. Therefore

  $$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

  Now

Cont./C.I. for μ

$$\begin{aligned} 1-\alpha &= P(-z_{1-\alpha/2} \le Z \le z_{1-\alpha/2}) \\ &= P(-z_{1-\alpha/2} \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le z_{1-\alpha/2}) \\ &= P(\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}) \end{aligned}$$

$(1-\alpha)100\%$ C.I for μ is

$$\left(\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

Or we can say $(1-\alpha)100\%$ C.I for μ is

$$\overline{X} \pm z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Clearly the form of the C.I. for μ is

$$Estimator \pm (Cut-of-point)\ S.E.(Estimator).$$

Cont./C.I. for μ

Example: Given a sample of 100 observations(temperatures) with $\sigma = 5$. Suppose the sample mean is $\overline{X} = 36$. Give a 90% C.I. for μ. It is in the form:

$$(36 - 1.64\frac{5}{10}, 36 + 1.64\frac{5}{10}) = (35.18, 36.82)$$

(A) Large Sample ($n \geq 30$) C. I. for μ (σ unknown):

$$\left(\overline{X} - z_{1-\alpha/2}\frac{S}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{S}{\sqrt{n}}\right)$$

(B) Small Sample ($n < 30$) C. I. for μ (σ unknown):

$$\left(\overline{X} - t_{\alpha/2}\frac{S}{\sqrt{n}}, \overline{X} + t_{\alpha/2}\frac{S}{\sqrt{n}}\right)$$

Note the degrees of freedom needed for t-distribution=n-1.

Cont./ Example

- **Example:** A sample of size 5 birth weights are taken from the normal data. These data are (in lb) $5, 6, 6.5, 5.8, 6.7$. Give a 90% C.I. for μ.

$$\overline{X} = 6, \quad Std. = S = 0.67$$

a 90% C.I. for μ is

$$\overline{X} \pm t_{0.05} \frac{S}{\sqrt{n}} = 6 \pm 2.13(\frac{0.67}{2.24}) = (5.36, 6.64)$$

- For large n, $\hat{p} \sim N(p, pq/n)$. Then $(1-\alpha)100\%$ C.I. for p is

$$\hat{p} \pm z_{1-\alpha/2} \sqrt{\frac{\hat{p}\hat{q}}{n}}$$

- Consider a sample of size n=100 voters produced $X = 60$ voters in favor of candidate A. Provide a 90% C.I. for p

$$0.6 \pm 1.96 \sqrt{\frac{(0.6)(0.4)}{100}} = 0.6 \pm (1.96)(0.049) = (0.504, 0.696).$$

Testing for μ

(A) Large Sample Test for μ Null hypothesis: $H_0 : \mu = \mu_0$
Alternative hypothesis: $H_1 : \mu > \mu_0 (\mu < \mu_0)$ *or* $H_1 : \mu \neq \mu_0$.

Test Statistic:

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}.$$

Rejection Region for one-sided hypotheses:

$$Z > z_{1-\alpha} \; (Z < -z_{1-\alpha})$$

Rejection Region for two-sided hypotheses:

$$Z > z_{1-\alpha/2} \;\; or \;\; Z < -z_{1-\alpha/2}.$$

Example: The blood pressure is taken to 64 persons. The sample mean is 135 mm Hg and Std. 20 mm Hg. Test at $\alpha = 0.05$ that
(a) $H_0 : \mu = 130$ vs. $H_1 : \mu > 130$
(b) $H_0 : \mu = 130$ vs. $H_1 : \mu \neq 130$

Example

Solution:

(a) $H_0 : \mu = 130$ vs. $H_1 : \mu > 130$

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = \frac{135 - 130}{20/8} = 2$$

Since observed $Z = 2 > z_{0.95} = 1.64$, we reject H_0.

(b) $H_0 : \mu = 130$ vs. $H_1 : \mu \neq 130$

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = \frac{135 - 130}{20/8} = 2$$

We reject H_0 if $Z < -1.96$ or $Z > 1.96$. Since observed $Z = 2 > z_{0.975} = 1.96$, we reject H_0.

Testing for p

(A) Large Sample Test for p Null hypothesis: $H_0 : p = p_0$
Alternative hypothesis: $H_1 : p > p_0 (p < p_0)\ \ or\ \ H_1 : p \neq p_0$. **Test Statistic:**

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0\ q_p}{n}}}.$$

Rejection Region for one-sided hypotheses:

$$Z > z_{1-\alpha}\ (Z < -z_{1-\alpha})$$

Rejection Region for two-sided hypotheses:

$$Z > z_{1-\alpha/2}\ \ or\ \ Z < -z_{1-\alpha/2}.$$

Cont./Example

Example (Breast Cancer Data): Out of 1000 mothers, 30 had breast cancer. Based on large studies, about 2% represent the rate of cancer. Do the above data present sufficient evidence to indicate that the rate of cancer is different from the well-known rate (2%). Use $\alpha = 0.05$.

- $H_0 : p = 0.02$ vs. $H_1 : p \neq 0.02$

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \ q_p}{n}}} = \frac{0.03 - 0.02}{\sqrt{\frac{(0.02)(0.98)}{100}}} = 2.26 > 1.96.$$

We reject H_0.

- To compute p-value, we have

p-value=2 (Area to the right of 2.26)=2(0.0119)=0.0238.

We reject H_0 based on the p-value.

Con./Example

- **Question:** Do the data present sufficient evidence that the cancer rate is greater than 0.02.

$H_0 : p = 0.02$ vs. $H_1 : p > 0.02$

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \ q_p}{n}}} = \frac{0.03 - 0.02}{\sqrt{\frac{(0.02)(0.98)}{100}}} = 2.26 > 1.64.$$

We reject H_0.

Determination of the Sample Size

- The sample size n for estimating μ. $(1-\alpha)100\%$ C. I. for μ is

$$\overline{X} \pm z_{1-\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

The error of estimation=$E = z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}$. Therefore

$$n = \left(\frac{z_{1-\alpha/2}}{E}\right)^2 \ \sigma^2.$$

- A researcher wants to estimate the average weight loss of people who are on a new diet plan. If $\sigma = 5$, how large a sample should be to estimate the mean weight loss by 95% to C.I. to within 1.5 kgs.?

$$n = \left(\frac{1.96}{1.5}\right)^2 \ 25 = 42.68 \approx 43.$$

Sample Size for estimating p

- The sample size n for estimating p. $(1-\alpha)100\%$ C. I. for p is

$$\hat{p} \pm z_{1-\alpha/2}.\sqrt{\hat{p}\,\hat{q}/n}$$

The error of estimation=$E = z_{1-\alpha/2}\sqrt{\hat{p}\,\hat{q}/n}$. Therefore

$$n = \left(\frac{z_{1-\alpha/2}}{E}\right)^2 \hat{p}\,\hat{q}.$$

We have two choices
(a) If no prior information about p is available, we choose $\hat{p} = 0.5$. In this case,

$$n = \frac{1}{4}\left(\frac{z_{1-\alpha/2}}{E}\right)^2.$$

(b) If there is an idea about p, say $p = p^*$. In this case,

$$n = \left(\frac{z_{1-\alpha/2}}{E}\right)^2 \; p^*(1-p^*).$$

Example

Example: It is required to estimate the proportion of patients suffering a bad reaction from taking a certain medication p by 95% C.I.. Determine the sample size needed if the error of estimation is about 0.10 in the following cases:

(a) no prior information about p

(b) previous study showed that $p \approx 0.20$.

Solution:

(a) no prior information about p

$$n = \left(\frac{1.96}{0.1}\right)^2 (0.25) \approx 97.$$

(b) previous study showed that $p \approx 0.20$.

$$n = \left(\frac{1.96}{0.1}\right)^2 (0.20)(0.80) \approx 62.$$

Inference about σ^2

- Remember that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$
- $(1-\alpha)100\%$ C.I. for σ^2 can be described as

$$\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}} \right)$$

- Example: $X_1, X_2, ..., X_{10} \sim N(\mu, \sigma^2)$ such that $S^2 = 10$. Find 90% C.I. of σ^2.

$$\left(\frac{9(10)}{16.92}, \frac{9(10)}{3.325} \right) = (5.32, 27.07)$$

Testing about σ^2

Null hypothesis: $H_0 : \sigma^2 = \sigma_0^2$
Alternative hypothesis: $H_1 : \sigma^2 > \sigma_0^2 (\sigma^2 < \sigma_0^2) \ \ or \ \ H_1 : \sigma^2 \neq \sigma_0^2$.

Test Statistic:

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}.$$

Rejection Region for one-sided hypotheses:

$$\chi^2 > \chi^2_\alpha \ (\chi^2 < \chi^2_{1-\alpha})$$

Rejection Region for two-sided hypotheses:

$$\chi^2 > \chi^2_{\alpha/2} \ \ or \ \ \chi^2 < \chi^2_{1-\alpha/2}.$$

Example:

$X_1, X_2, ..., X_{10} \sim N(\mu, \sigma^2)$ such that $S^2 = 10$. Test $H_0 : \sigma^2 = 8$ vs. $H_1 : \sigma^2 > 8$.

Test Statistic:

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} = \frac{90}{8} = 11.25.$$

Since $\chi^2 = 11.2 \not> \chi^2_{0.05} = 16.92$, we accept H_0 under $\alpha = 0.05$.

For $H_0 : \sigma^2 = 8$ vs. $H_1 : \sigma^2 \neq 8$.

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} = \frac{90}{8} = 11.25.$$

Since $\chi^2 = 11.2 \not> \chi^2_{0.025} = 19.02$, we accept H_0 under $\alpha = 0.05$.