Inference about 2 populations

Comparing 2 populations means

(A)Inference for large samples.

Assume $X_1, X_2, ..., X_m \sim N(\mu_1, \sigma_1^2)$ and $Y_1, Y_2, ..., Y_m \sim N(\mu_2, \sigma_2^2)$. Then

$$\overline{X} \sim N(\mu_1, \sigma_1^2/m), \;\; \overline{Y} \sim N(\mu_2, \sigma_2^2/n)$$

Consequently,

$$\frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \sim N(0, 1)$$

(I) $100(1 - \alpha)\%$ C.I. for $\mu_1 - \mu_2$ is

$$(\overline{X} - \overline{Y}) \pm z_{1-\alpha/2}\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}$$

Testing about $\mu_1 - \mu_2$

(II) Testing Null hypothesis: $H_0 : \mu_1 - \mu_2 = 0$
Alternative hypothesis: $H_1 : \mu_1 - \mu_2 > 0 (\mu_1 - \mu_2 < 0) \;\; or \;\; H_1 : \mu_1 - \mu_2 \neq 0$.

Test Statistic:

$$Z = \frac{(\overline{X} - \overline{Y}) - 0}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}} \sim N(0,1)$$

Rejection Region for one-sided hypotheses:

$$Z > z_{1-\alpha} \; (Z < -z_{1-\alpha})$$

Rejection Region for two-sided hypotheses:

$$Z > z_{1-\alpha/2} \;\; or \;\; Z < -z_{1-\alpha/2}.$$

Example on Inference about $\mu_1 - \mu_2$

Example: To compare the quality of two types of tires, samples of m=n=100 tires are taken with the following information.

	Mean	Std.	Sample size
Tire A	26,000	1200	100
Tire B	25,000	1400	100

(a) Give 99% C.I. for $\mu_1 - \mu_2$

$$\begin{aligned}(26,000 - 25,000) \pm (2.58)\sqrt{\frac{1440000}{100} + \frac{1960000}{100}} &= 1000 \pm (2.58)(184) \\ &= (525.28, 1474.72)\end{aligned}$$

(b)Do the data provide strong evidence that Tire A is different from Tire B?
$H_0 : \mu_1 - \mu_2 = 0$ vs. $H_1 : \mu_1 - \mu_2 \neq 0$.

Since $Z = 1000/184 = 5.43 > z_{0.975} = 1.96$, we reject H_0.

Comparing 2 means for small samples

(A)Inference for small samples.

- Assume $X_1, X_2, ..., X_m \sim N(\mu_1, \sigma_1^2)$ and $Y_1, Y_2, ..., Y_m \sim N(\mu_2, \sigma_2^2)$. Here we assume $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Then the pooled estimate of σ is

$$S_p = \sqrt{\frac{(m-1)S_1^2 + (n-1)S_2^2}{m+n-2}}$$

- The test statistic is

$$\frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{m} + \frac{1}{n}}} \sim t_{m+n-2}$$

(I) $100(1-\alpha)\%$ C.I. for $\mu_1 - \mu_2$ is

$$(\overline{X} - \overline{Y}) \pm t_{\alpha/2} S_p \sqrt{\frac{1}{m} + \frac{1}{n}}$$

Testing about $\mu_1 - \mu_2$

(II) Testing Null hypothesis: $H_0 : \mu_1 - \mu_2 = 0$
Alternative hypothesis: $H_1 : \mu_1 - \mu_2 > 0 (\mu_1 - \mu_2 < 0) \;\; or \;\; H_1 : \mu_1 - \mu_2 \neq 0$.

Test Statistic:

$$t = \frac{(\overline{X} - \overline{Y}) - 0}{S_p \sqrt{\frac{1}{m} + \frac{1}{n}}}$$

Rejection Region for one-sided hypotheses:

$$t > t_{1-\alpha} \; (t < -t_{1-\alpha})$$

Rejection Region for two-sided hypotheses:

$$t > t_{\alpha/2} \;\; or \;\; t < -t_{\alpha/2}.$$

Note the degrees of freedom is $df = m + n - 2$

Example on Inference about $\mu_1 - \mu_2$

Example: Given the following table:

	Mean	Std.	Sample size
Treatment 1 A	45	5	25
Treatment 2	40	4	16

(a) Give 95% C.I. for $\mu_1 - \mu_2$

The pooled estimate of σ is

$$S_p = \sqrt{\frac{(24)(25) + (15)(16)}{39}} = \sqrt{\frac{881}{39}} = 4.64$$

$$(45 - 40) \pm (2.02)(4.64)\sqrt{0.0625 + 0.04} = 5 \pm (2.02)(4.64)(0.32) = (2.22, 7.88)$$

(b)Test at $\alpha = 0.025$ that the mean of Trt.2 is less than that of Trt.1?

$H_0 : \mu_1 - \mu_2 = 0$ vs. $H_1 : \mu_1 - \mu_2 > 0$.

Since $t = 5/\left[(4.64)\sqrt{\frac{1}{25} + \frac{1}{16}}\right] = 3.38 > t_{0.025} = 2.021$, we reject H_0.

Paired Data

- Two samples are said to be paired when each data point in the first sample is matched and related to a unique data point of the second sample.

1_{st} reading	2_{nd} reading	difference
X_1	Y_1	$d_1 = X_1 - Y_1$
X_2	Y_2	$d_2 = X_2 - Y_2$
–	–	–
X_n	Y_n	$d_n = X_n - Y_n$

- The sample mean and standard deviation of d_i's are

$$\bar{d} = \frac{\sum_{i=1}^{n} d_i}{n} \quad \text{and} \quad S_d^2 = \frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1}.$$

Cont. Paired data

- Assume the d_i's are normally distributed with mean μ_d and variance σ_d^2.
  (I) $100(1-\alpha)\%$ C. I. for μ_d is

  $$\bar{d} \pm t_{\alpha/2} \frac{S_d}{\sqrt{n}}$$

  (II) Testing about μ_d
  Test Statistic:

  $$\frac{\bar{d} - \mu_d}{S_d/\sqrt{n}} \sim t_{n-1}$$

  Rejection Region for one-sided hypotheses:

  $$t > t_\alpha \ (t < -t_\alpha)$$

  Rejection Region for two-sided hypotheses:

  $$t > t_{\alpha/2} \ \ or \ \ t < -t_{\alpha/2}.$$

  Note the degrees of freedom is $df = n - 1$

Example on paired data

Example: One method for assessing the effectiveness of a drug is to note concentration in blood and/or urine samples.

Person	Before medication	After medication	d_i
1	15	13	2
2	26	20	6
3	13	10	3
4	28	21	7
5	17	17	0
6	20	22	-2
7	7	5	2
8	36	30	6
9	12	7	5
10	18	11	7
mean	19.2	15.6	3.6
std.	8.63	7.78	3.098

Cont. Example

(a) Give 95% C.I. for μ_d

$$\overline{d} \pm t(0.025)\frac{S_d}{n} = 3.6 \pm (2.262)\frac{3.098}{\sqrt{10}} = (1.38, 5.82)$$

(b)Use $\alpha = 0.01$ to test $H_0 : \mu_d = 0$ vs. $H_1 : \mu_d \neq 0$.
Since

$$t = \frac{\overline{d}}{S_d/\sqrt{n}} = \frac{3.6}{3.09/\sqrt{10}} = 3.67 > t_{0.005} = 3.25$$

we reject H_0.

Comparing 2 proportions

- Assume $X = \#$ of successes in m trials from Pop.1 and $Y = \#$ of successes in n trials from Pop.2. Therefore

$$\hat{p_1} = \frac{X}{m}, \quad \hat{p_2} = \frac{Y}{n}.$$

- $\hat{p}_1 - \hat{p}_2 \sim N(p_1 - p_2, \frac{p_1\ q_1}{m} + \frac{p_2\ q_2}{n})$
- (I) $100(1-\alpha)\%$ C.I. for $p_1 - p_2$ is

$$(\hat{p}_1 - \hat{p}_2) \pm z_{1-\alpha/2} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n}}$$

- The test statistic is

$$\frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}(1-\hat{p})}\sqrt{\frac{1}{m} + \frac{1}{n}}}$$

where $\hat{p} = (X+Y)/(m+n)$ is the pooled estimate for $p = p_1 = p_2$.

Example on Inference about $p_1 - p_2$

Example: Given the following table:

	Sample size	# Support space research
Republican	64	45
Democrat	100	60

(a) Give 80% C.I. for $p_1 - p_2$=difference in proportions of the people who are supporting the space research

$$(0.7 - 0.6) \pm (1.28)\sqrt{\frac{(0.7)(0.3)}{64} + \frac{(0.6)(0.4)}{100}} = 0.1 \pm (1.28)(0.075) = (0.004, 0.196)$$

(b)Test at $\alpha = 0.01$ that the proportion of people who supporting space research from Republican is greater than that of democrat?

$H_0 : p_1 - p_2 = 0$ vs. $H_1 : p_1 - p_2 > 0$.

The pooled estimate of common p is $\hat{p} = (45 + 60)/164 = 0.64$. Since

$$Z = \frac{(0.7 - 0.6) - 0}{0.048\sqrt{\frac{1}{64} + \frac{1}{100}}} = 1.3 \not> z_{0.99} = 2.33,$$

we accept H_0.

Comparing 2 variances

- Assume $X_1, X_2, ..., X_m \sim N(\mu_1, \sigma_1^2)$ and $Y_1, Y_2, ..., Y_m \sim N(\mu_2, \sigma_2^2)$ are independent samples. Let S_1^2 and S_2^2 be the sample variances of Pop.1 and 2, respectively.

- $(m-1)S_1^2/\sigma_1^2 \sim \chi^2_{m-1}$ and $(n-1)S_2^2/\sigma_2^2 \sim \chi^2_{n-1}$. In a consequence of that,

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(m-1, n-1)$$

  where $F(m-1, n-1)$ is Fisher distribution with degrees of freedom $m-1$ and $n-1$.

- If $F \sim F(\nu_1, \nu_2)$ then $\frac{1}{F} \sim F(\nu_2, \nu_1)$. This means that $F_{0.95}(\nu_1, \nu_2) = \frac{1}{F_{0.05}(\nu_2, \nu_1)}$

- (I) $100(1-\alpha)\%$ C.I. for σ_1^2/σ_2^2 is

$$\left(\frac{S_1^2/S_2^2}{F_{\alpha/2}(m-1, n-1)}, \frac{S_1^2/S_2^2}{F_{1-\alpha/2}(m-1, n-1)} \right)$$

Testing for 2 variances

(II) Testing about σ_1^2/σ_2^2 Testing Null hypothesis: $H_0 : \sigma_1^2/\sigma_2^2 = 1$
Alternative hypothesis: $H_1 : \sigma_1^2/\sigma_2^2 > 1(\sigma_1^2/\sigma_2^2 < 1)\ \ or\ \ H_1 : \sigma_1^2/\sigma_2^2 \neq 1$.

Test Statistic:

$$F = \frac{S_1^2}{S_2^2}$$

Rejection Region for one-sided hypotheses:

$$F > F_\alpha\ (F < F_{1-\alpha})$$

Rejection Region for two-sided hypotheses:

$$F > F_{\alpha/2}\ \ or\ \ F < F_{1-\alpha/2}.$$

Note the degrees of freedom is $df = (m-1, n-1)$

Example on Inference about σ_1^2/σ_2^2

Example: Given the following table:

	Sample size	Std.
Section 1	16	5
Section 2	9	3

(a) Give 90% C.I. for σ_1^2/σ_2^2 = ratio of the two populations variances

$$\left(\frac{25/9}{F_{0.05}(15,8)}, \frac{25/9}{F_{0.95}(15,8)}\right) = (\frac{2.78}{3.22}, (2.78)(3.22)) = (0.86, 7.34)$$

(b)Test

$H_0 : \sigma_1^2/\sigma_2^2 = 1$ vs. $H_1 : \sigma_1^2/\sigma_2^2 > 1$.

$$F = \frac{25}{9} = 2.78 < F_{0.05}(15,8) = 3.22,$$

we accept H_0.