Combination Technique

- Often the order of selection is not important and interest only on the selected set of r objects from a set of N distinct objects. That is, we are interested in the number of subsets of size r that can be selected from a set of N distinct objects. This number is called N combination r or the number of combinations that are possible when selecting r objects from a set of N distinct objects.

  The number of combinations of N distinct objects chosen r at a time is

$$\binom{N}{r} =_N C_r = \frac{N!}{r!(N-r)!}, \; 0 \leq r \leq N$$

  Note that $\binom{N}{r}$ is called binomial coefficient.

Example: A lot consists of 1100 distinct items. There are 4 percent defective items in the lot. What is the probability that a random sample of size 50 items contains 4 defective items?

Solution: Let A be the event that the random sample of size 50 items contains 4 defective items. In the lot there are $(4/100)(1100) = 44$ defective items and $1100 - 44 = 1056$ non-defective items.
There are $\binom{1100}{50}$ total possible outcomes. Also there are $\binom{44}{4}$ ways of choosing the 4 defective items from the 44 defective items and $\binom{1056}{46}$ ways of choosing the remaining 46 non-defective items from the 1056 non-defective items. Hence

$$P(A) = \frac{\binom{44}{4}\binom{1056}{46}}{\binom{1100}{50}}$$

Axiomatic Probability: If the sample space of a random experiment is not finite, or the outcomes of the sample space are not equally likely, we cannot use the classical definition of probability. To solve this problem we consider the axiomatic definition of probability.

The probability function, P, is a real-valued set function whose domain is a collection of all events in a sample space S that satisfies the following axioms:

(1) For every event A, $P(A) \geq 0$
(2) $P(S) = 1$
(3) If $A_1, A_2, ..., A_n, ...$ is a sequence of mutually exclusive events, that is, $A_i \bigcap A_j = \phi$ for all $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Probability Rules:

(1) $P(\phi) = 0$

(2) $P(\overline{A}) = 1 - P(A)$

Example: If a pair of dice is rolled one time. Find the probability that the sum is at most 11. Assuming that all outcomes are equally likely.

Solution: The sample space for this experiment is given by:

$$S = \{(x, y) : x, y = 1, 2, 3, 4, 5, 6\}$$

Let A be the event that the sum is at most 11. Then $\overline{A}$ represent the sum is greater than 11, that is, 12. Hence $\overline{A} = (6, 6)$ and

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{n(\overline{A})}{n(S)} = 1 - \frac{1}{36} = 0.972.$$

Cont./Probability Rules:

(3) Additive law of probability: $P(A \bigcup B) = P(A) + P(B)$ if A and B disjoint (mutually exclusive).

(4) Multiplicative law of probability: $P(A \bigcap B) = P(A).P(B)$ if A and B independent. In general,

$$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$$

Example: Two detectors of smoking A and B are working independently with

P(A)=P(1st one detects is smoking)=0.6
and
P(B)=P(2nd one detects smoking)=0.5.

Find the probability that at least one detector is working.

Solution:

- The requested probability is just $P(A \bigcup B)$, which is

$$\begin{aligned} P(A \bigcup B) &= P(A) + P(B) - P(A \bigcap B) \\ &= 0.6 + 0.5 - (0.6)(0.5) \\ &= 0.8 \end{aligned}$$

- One can also find the probability as
  P(at least one is working)=1- P(none is working). That is;
  P(at least one is working)= 1- P(1st is not working)P(2nd is not working)=1-(0.4)(0.5)=0.8

Example: Urn 1 contains two white balls and one black ball. Urn 2 contains one white ball. A ball is drawn from Urn 1 and placed in Urn 2. Then a ball is drawn from Urn 2. What is the probability that the ball drawn from Urn 2 will be white?

Event	Urn 1	Urn 2	$P(E_i)$
E_1	W_1	W_1	$\frac{1}{6}$
E_2	W_1	W_3	$\frac{1}{6}$
E_3	W_2	W_2	$\frac{1}{6}$
E_4	W_2	W_3	$\frac{1}{6}$
E_5	B_1	B_1	$\frac{1}{6}$
E_6	B_1	W_3	$\frac{1}{6}$

P(2nd ball is white)=5/6.

Conditional Probability:

Let A: observe rain and B: Observe cloudy sky. Note that A and B are obviously related. Then $P(A|B)$ = Conditional probability of A given B has occurred= Probability

that we observe rain given that it was cloudy. We have

$$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$$

Consequently

$$\begin{aligned} P(A \bigcap B) &= P(A|B)P(B) \\ &= P(B|A)P(A) \end{aligned}$$

Previous Example:

A=ball drawn from Urn 2 is white

B= draw a white ball from Urn 1 and a white ball from Urn 2

C=draw a black ball from Urn 1 and a white ball from Urn 2

Then

$$P(A) = P(B \bigcup C) = P(B) + P(C)$$

Note that

$$B = B_1 \bigcap A$$

and

$$C = C_1 \bigcap A$$

where B_1 is a white ball from Urn 1 and C_1 is a black ball from Urn 1.

Now $P(B) = P(B_1 \bigcap A) = P(A|B_1)P(B_1) = 1(2/3) = 2/3$

and $P(C) = P(C_1 \bigcap A) = P(A|C_1)P(C_1) = (1/2)(1/3) = 1/6$

Therefore

$$P(A) = \frac{2}{3} + \frac{1}{6} = \frac{5}{6}$$

Additive law of probability:

The probability law can be written as

$$\begin{aligned} P(A\bigcup B) &= P(A)+P(B)-P(A\bigcap B) \\ &= P(A)+P(B)-P(A|B)P(B) \\ &= P(A)+P(B)-P(B|A)P(A) \end{aligned}$$

If A and B are independent, then

$$P(A\bigcup B)=P(A)+P(B)-P(A)P(B)$$

If A and B are disjoint, then $P(A\bigcap B)=\phi$ and

$$P(A\bigcup B)=P(A)+P(B)$$