Bayes Theorem

- (1) Partitioning the sample space into $B_1, B_2, ..., B_k$ disjoint events $(P(B_i \bigcap B_j) = 0$ for $i \neq j$.

  (2) Given $P(A|B_1), ..., P(A|B_k)$ which represent prior probabilities

  (3) Required: $P(B_i|A)$

Byes Theorem:

$$P(B_i|A) = \frac{P(B_i \bigcap A)}{P(A)}$$

$$= \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{k} P(A \bigcap B_i)}$$

$$= \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}$$

Example: The mails in US are delivered by three companies FAD, CAD and NAD with

$$P(FAD) = 0.4, P(CAD) = 0.3, P(NAD) = 0.3$$

Also

$$P(Late|FAD) = 0.2, P(Late|CAD) = 0.4, P(Late|NAD) = 0.5$$

(1) Given the mail is late, what is the probability that it was delivered by FAD.

$$\begin{aligned} P(FAD|L) &= \frac{0.08}{0.08 + 0.12 + 0.15} \\ &= \frac{0.08}{0.35} \\ &= 0.23 \end{aligned}$$

Cont./ Example:

(2) Given the mail is late, what is the probability that it was delivered by FAD or NAD.

$$\begin{aligned} P(FAD|L) + P(NAD|L) &= \frac{0.08 + 0.15}{0.08 + 0.12 + 0.15} \\ &= \frac{0.23}{0.35} \\ &= 0.66 \end{aligned}$$

(3) What is the probability that it was late?

$$P(Late) = 0.08 + 0.12 + 0.15 = 0.35$$

(4) Given the mail is not late, What is the probability that it was delivered by CAD?

$$P(CAD|\overline{L}) = \frac{0.18}{0.65} = 0.28$$

Example:

Box I contains 3 Red and 2 Black balls
Box II contains 1 Red and 4 Black balls

If 2 balls are transferred from Box I into Box II and then one ball is drawn from Box II. If the ball drawn from Box II was red, what is the probability the balls transferred from Box I are both red.

Solution: Here one can use the tree argument (what is that?).

$$P(\text{Balls transferred from Box I are red}|\text{Ball drawn from II was red}) = \frac{9/70}{(9+12+1)/70} = \frac{9}{22}$$

Example: If

$$P(A \bigcap \overline{B}) = 0.4, \;\; P(\overline{A} \bigcap B) = 0.2, \;\; \text{and} P(\overline{A} \bigcap \overline{B}) = 0.3.$$

Find

(i) $P(A)$

(ii)$P(A)$

(iii) $P(\overline{A}|B)$

Solution: From $P(\overline{A} \bigcap \overline{B}) = 0.3$, we have $P(A \bigcup B) = 0.7$. Now

(i) $P(A \bigcap B) = 0.1$ and then

$$\begin{aligned} P(A \bigcap \overline{B}) &= P(A) - P(A \bigcap B) \\ 0.4 &= P(A) - 0.1 \end{aligned}$$

Then $P(A) = 0.5$.

Cont./Example: (ii) $P(A \bigcap B) = 0.1$ and then

$$\begin{aligned} P(\overline{A} \bigcap B) &= P(B) - P(A \bigcap B) \\ 0.2 &= P(B) - 0.1 \end{aligned}$$

Then $P(A) = 0.3$.

(ii) $P(\overline{A}|B)$

$$\begin{aligned} P(\overline{A}|B) &= \frac{P(\overline{A} \bigcap B)}{P(B)} \\ &= \frac{0.2}{0.3} \\ &= \frac{2}{3} \end{aligned}$$

Example:

Out of 12 people applying for a job, 3 cannot do the work. Suppose that 2 persons will be hired.

(a) In how many ways will 0 or 1 people not be able to do the work

(b)What is the probability that 0 or 1 people not be able to do the work

(c) If 2 persons are chosen in random, what is the probability that neither will be able to do the job.

Solution:

(a) In how many ways will 0 or 1 people not be able to do the work

$$\begin{aligned} &= \binom{3}{0}\binom{9}{2} + \binom{3}{1}\binom{9}{1} \\ &= \frac{(9)(8)}{(1)(2)} + (3)(9) \\ &= 36 + 27 \\ &= 63 \end{aligned}$$

Cont./ Example: (b)What is the probability that 0 or 1 people not be able to do the work

$$\begin{aligned} &= P(X=0)+P(X=1) \\ &= \frac{\binom{3}{0}\binom{9}{2}+\binom{3}{1}\binom{9}{1}}{\binom{12}{2}} \\ &= \frac{63}{66} \\ &= \frac{21}{22} \end{aligned}$$

(c) If 2 persons are chosen in random, what is the probability that neither will be able to do the job.

$$\begin{aligned} &= \frac{\binom{3}{2}\binom{9}{0}}{\binom{12}{2}} \\ &= \frac{1}{22} \end{aligned}$$

Example:

For each of 20 questions on a multiple-choice test, a student can choose one of 5 answers

(a) How many different sets of answers are possible?

(b) If a person guessed on all the questions, what is the probability that all of the questions would be answered correctly?

Answer:

(a) How many different sets of answers are possible?

$$\text{answer} = 5^{20}$$

(b) If a person guessed on all the questions, what is the probability that all of the questions would be answered correctly?

$$\text{Probability} = \frac{1}{5^{20}}$$