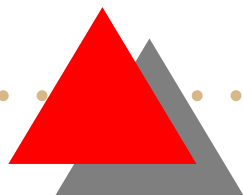


Random Variables and Probability Distribution





Definitions:

- **Random Variable:** A random variable is a numerically valued function defined over a sample space S .
- **Discrete Random Variable:** It is one that can assume a countable number of values.

Examples:

(1) # of defective items drawn from a sample of size 10.

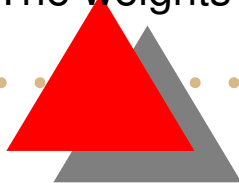
(2) # of people in the waiting line in a doctor's office from a sample of size 100.

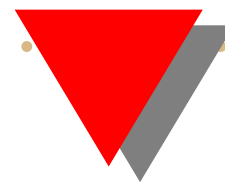
- **Continuous Random Variable:** It is one that can assume infinitely large number of values.

Examples:

(1) The failure times of electric bulbs

(2) The weights of children entering the doctor's office





Probability Distribution for Discrete R.V.'s:

$$0 \leq p(x) \leq 1$$

$$\sum_x p(x) = 1$$

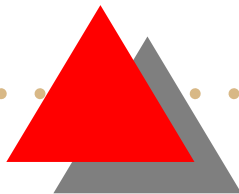
Example: Tossing a coin twice. Let X = number of heads=0,1,2.

$$P(X = 0) = \frac{1}{4}$$

$$P(X = 1) = \frac{1}{2}$$

$$P(X = 2) = \frac{1}{4}.$$

X	0	1	2
$p(x)$	1/4	1/2	1/4





Example: Out of 5 items, 3 items are defectives. Two items are drawn at random. Let X = number of defective items chosen. That is, $X = 0, 1, 2$.

$$P(X = 0) = \frac{\binom{3}{0} \binom{2}{2}}{\binom{5}{2}} = 0.1$$

$$P(X = 1) = \frac{\binom{3}{1} \binom{2}{1}}{\binom{5}{2}} = 0.6$$

$$P(X = 2) = \frac{\binom{3}{2} \binom{2}{0}}{\binom{5}{2}} = 0.3$$

X	0	1	2
$p(x)$	0.1	0.6	0.3





Example: Assume $p = P(S) = (\text{success}) = 0.4$, $q = P(F) = P(\text{Failure})$, and $X =$ number of drillings until the first success occurs. So $X = 1, 2, 3, \dots$

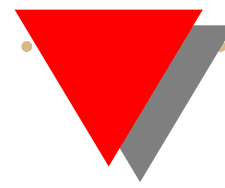
$$P(X = 1) = P(S) = 0.4$$

$$P(X = 2) = P(FS) = (0.6)(0.4) = 0.24$$

$$P(X = 3) = P(FFS) = (0.6)(0.6)(0.4) = 0.144$$

X	$P(X)$
1	0.4
2	0.24
3	0.144
...	





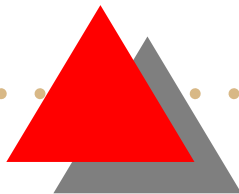
Expectation and Its Properties:

X	$P(x)$	$xP(x)$	$x^2P(x)$
0	0.1	0	0
1	0.6	0.6	0.6
2	0.3	0.6	1.2
		1.2	1.8

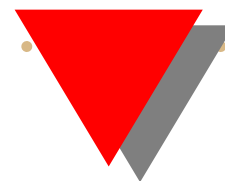
$$EX = \sum_x x P(x) = 1.2, \text{ and } EX^2 = \sum_x x^2 P(x) = 1.8$$

$$\begin{aligned} \sigma^2 = Var(X) &= E(X - \mu)^2 = EX^2 - \mu^2 \\ &= 1.8 - (1.2)^2 = 0.36. \end{aligned}$$

and



$$\sigma = \sqrt{0.36} = 0.6$$



Properties on Expectation:

(i) $E(X + a) = E(X) + a$

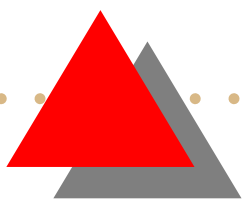
(ii) $E(a + bX) = a + bE(X)$

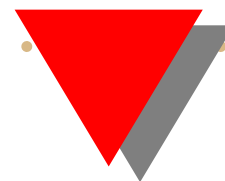
(iii) $E(a + bX + cX^2) = a + bE(X) + cE(X^2)$

(iv) $Var(X + a) = Var(X)$

(v) $Var(aX + b) = a^2 Var(X)$

(vi) $Std(aX + b) = |a|Std(X)$.



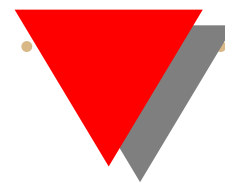


Example: A sales person receives orders for 0, 1, 2, 3 or 4 units of products each day with probability 0.4, 0.1, 0.3, 0.1, and 0.1, respectively. His salary is 50 JD per day plus 10 JD for each unit ordered.

X	$P(x)$	$xP(x)$	$x^2P(x)$
0	0.4	0	0
1	0.1	0.1	0.1
2	0.3	0.6	1.2
3	0.1	0.3	0.9
4	0.1	0.4	1.6
		1.4	3.8

His expected salary is $S = 50 + 10E(X) = 50 + 14 = 64$.





Joint Distribution of 2 R.V.'s:

Let

$$X = x_1, x_2, \dots, x_k$$

$$Y = y_1, y_2, \dots, y_l,$$

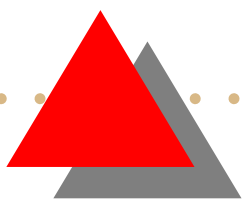
with

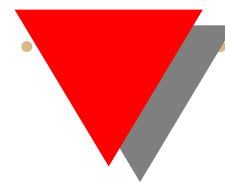
$$f(x_i, y_j) = P(X = x_i, Y = y_j)$$

Suppose there are 10 cars in a lot of which 5 are in good condition, 2 have defective transmission (DT), and the other 3 have defective steering (DS). 2 cars are chosen at random.

Define

$X = \#$ of cars with DT=0,1,2 and $Y = \#$ of cars with DS=0,1,2.





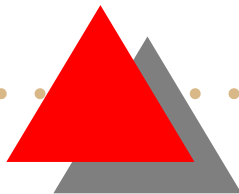
Cont./ Example:

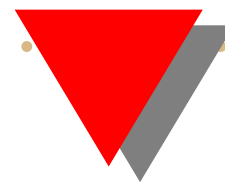
$$f(0, 0) = P(X = 0, Y = 0) = \frac{\binom{5}{2}}{\binom{10}{2}} = \frac{10}{45}$$

$$f(0, 1) = P(X = 0, Y = 1) = \frac{\binom{2}{0} \binom{3}{1} \binom{5}{1}}{\binom{10}{2}} = \frac{15}{45}$$

$$f(0, 2) = P(X = 0, Y = 2) = \frac{\binom{2}{0} \binom{3}{2} \binom{5}{0}}{\binom{10}{2}} = \frac{3}{45}$$

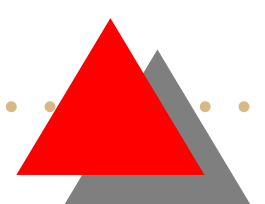
$$f(1, 2) = P(X = 1, Y = 2) = 0$$

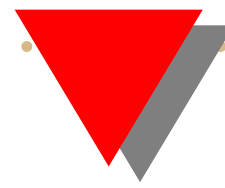




Cont./ Example:

		Y			
		0	1	2	
X	0	10/45	15/45	3/45	28/45
	1	10/45	6/45	0	16/45
	2	1/45	0	0	1/45
$P(y)$		21/45	21/45	3/45	1





Cont./ Example:



$$P(X = 0) = \frac{28}{45}$$

$$P(Y = 1) = \frac{21}{45}$$

$$P(X > Y) = f(1, 0) + f(2, 0) + f(2, 1) = \frac{10}{45} + \frac{1}{45} + 0 = \frac{11}{45}$$

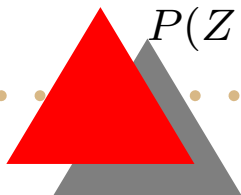
- Find the distribution of $Z = X + Y$.

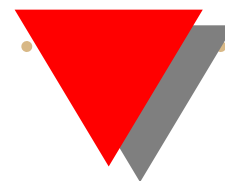
Now $Z = 0, 1, 2, 3, 4$.

$$P(Z = 0) = P(0, 0) = \frac{10}{45}, \quad P(Z = 1) = P(1, 0) + P(0, 1) = \frac{25}{45}$$

$$P(Z = 2) = P(0, 2) + P(2, 0) + P(1, 1) = \frac{10}{45}$$

$$P(Z = 3) = P(1, 2) + P(2, 1) = 0, \quad P(Z = 4) = P(2, 2) = 0.$$





Cont./ Example:

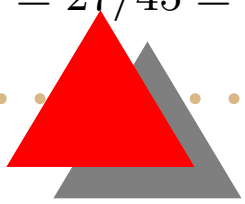
- Find the mean and Variance of X

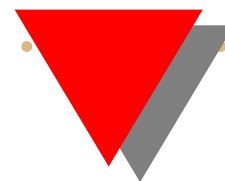
X	$P(x)$	$xP(x)$	$x^2P(x)$
0	28/45	0	0
1	16/45	16/45	16/45
2	1/45	2/45	4/45
		18/45	20/45

Therefore $EX = 18/45 = 0.4$, $Var(X) = (20/45) - (18/45)^2 = 0.44 - 0.16 = 0.284$ $\sigma_X = 0.53$.

Similarly,

$EY = 27/45 = 0.6$, $Var(X) = (33/45) - (27/45)^2 = 0.373$, $\sigma_Y = 0.61$.





Cont./ Example:

- Covariance and Correlation Coefficient X and Y . It is a measure of association. It is defined as

$$Cov(X, Y) = E(XY) - E(X).E(Y)$$

where $E(XY) = \sum_{x,y} (x y) p(x, y)$. The correlation coefficient between X and Y is

$$Corr(X, Y) = \rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

Then

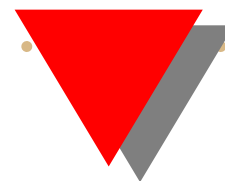
$$E(X Y) = (0)(0)(10/45) + (1)(1)(6/45) + (1)(2)(0) + \dots + = 6/45,$$

$$Cov(X, Y) = \frac{6}{45} - \frac{18}{45} \frac{27}{45} = 0.13 - 0.24 = -0.11$$

and

$$\rho = Corr(X, Y) = \frac{-0.11}{(0.53)(0.61)} = -0.33$$





Properties on Covariance and Correlation:

- $-1 \leq \rho \leq 1$
- $\text{Corr}(aX+b, cY+d) = \text{Corr}(X, Y)$ if a and c have the same signs and $\text{Corr}(aX+b, cY+d) = -\text{Corr}(X, Y)$ if a and c have opposite signs
- $\text{Var}(X \mp Y) = \text{Var}(X) + \text{Var}(Y) \mp 2\text{Cov}(X, Y)$
- $\text{Var}(aX \mp bY + c) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) \mp 2ab \text{Cov}(X, Y)$
- If X and Y are indep. with $EX = 0, \text{Std}(X) = 2, EY = -1, \text{Std}(Y) = 4$, then

$$\text{Var}(X - Y) = 4 + 16 = 20$$

$$\text{Var}(0.5X + 0.5Y) = (1/4)(4) + (1/4)(16) = 1 + 4 = 5$$

- If $\text{Cov}(X, Y) = 1$, then
 $\text{Var}(0.5X + 0.5Y) = (1/4)(4) + (1/4)(16) + 2(1/2)(1/2) = 5.5$

