Distributions of Discrete Random Variables

Binomial experiment:

(1) n identical trials

(2) Each trial results in one of two outcomes either success S or failure F

(3) Trials are independent

(4) We are interested in:

X= number of successes $(0, 1, ..., n)$

X	$p(x)$	
0	q^2	$\binom{2}{0} p^0 q^{2-0}$
1	$2pq$	$\binom{2}{1} p^1 q^1$
2	p^2	$\binom{2}{2} p^2 q^{2-2}$

In general $P(X = x) = \binom{n}{x} p^x q^{n-x}, x = 0, 1, ..., n$

Example: Assume that the probability of hitting a target=0.8. If a person fires 4 shots at the target. (a) What is the probability that he will hit the target exactly 2 times?

$$\begin{aligned} P(X=2) &= \binom{4}{2}(0.8)^2(0.2)^2 \\ &= 6(0.64)(0.04) \\ &= 0.1536 \end{aligned}$$

(b) What is the probability of hitting the target at least twice?

$$\begin{aligned} P(X \geq 2) &= 1 - P(0) - P(1) \\ &= 1 - \binom{4}{0}(0.8)^0(0.2)^4 - \binom{4}{1}(0.8)^1(0.2)^3 \\ &= 1 - 0.027 \\ &= 0.973 \end{aligned}$$

Binomial Tables: The tables presents $P(X \leq c)$ up to $n = 25$ with some different values of p.

$$P(X = c) = P(X \leq c) - P(X \leq c - 1)$$

$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a - 1)$$

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a)$$

$$P(X > c) = 1 - P(X \leq c)$$

Mean and Variance:

Mean = $\mu = np$, Variance=$\sigma^2 = npq$ and $\sigma = \sqrt{npq}$.

Hypergeometric Distribution:

$$P(X = x) = \frac{\binom{D}{x}\binom{D'}{n-x}}{\binom{N}{n}},$$

where $D' = N - D$ and $X = \max(0, n - D'), ..., \min(D, n)$.

Mean and Variance:

- Mean = $\frac{n\ D}{N}$
- Var(X)=$n\ \frac{D}{N}(1 - \frac{D}{N})(\frac{N-n}{N-1})$. Note the last term tends to be one if n is small with respect to N.

Example: A box is containing 3 black and 2 white balls. Two balls are drawn. Let $X =$ number of while balls.

- Drawing with replacement

$$P(X=0) = (\frac{3}{5})(\frac{3}{5}) = \frac{9}{25}$$

$$P(X=1) = 2\left(\frac{3}{5}\frac{2}{5}\right) = \frac{12}{25}$$

$$P(X=2) = (\frac{2}{5})(\frac{2}{5}) = \frac{4}{25}$$

In a general form,

$$P(X=x) = \binom{2}{x}(\frac{2}{5})^{x}(\frac{2}{5})^{2-x}, \; x = 0, 1, 2.$$

- Drawing without replacement

$$P(X=0) = (\frac{3}{5})(\frac{2}{4}) = \frac{3}{10}$$

$$P(X=1) = 2\left(\frac{3}{5}\frac{2}{4}\right) = \frac{6}{10}$$

$$P(X=2) = (\frac{2}{5})(\frac{1}{4}) = \frac{1}{10}$$

In a general form,

$$P(X=x) = \frac{\binom{2}{x}\binom{3}{2-x}}{\binom{5}{2}},\ x = 0, 1, 2.$$

Geometric Distribution:

Let $X = \#$ of trials to get the first success. Then

$$P(X = x) = q^{x-1}\ p,\ x = 1, 2, ...,$$

For example, continue examining n individuals until an affected person is found.

- Mean=$\frac{1}{p}$ and Var(X)=$\sigma^2 = \frac{q}{p^2}$.
- Keep in tossing 2 balanced dice until the sum of the points in 2 faces 7 appears. Let $X =$ number of tosses needed until. Note $p =$P(sum=7)=6/36=1/6, $\mu = 1/p = 6$ and $\sigma = \sqrt{q/p^2} = \sqrt{30} = 5.47$. The probability distribution of X is

$$P(X = x) = (\frac{5}{6})^{x-1}\ \frac{1}{6},\ x = 1, 2, ...,$$

$$\begin{aligned} P(X > 2\mu - 2\sigma) &= P(X > 1.06) = 1 - P(X = 1) \\ &= \frac{5}{6}. \end{aligned}$$

Poisson Distribution:

This distribution is called the distribution of rate events. Let $X =$ be the number of occurrences in a time interval.

- It depends only on the average rate of occurrences, this does not require the knowledge of n and p individually.
- Poisson distribution provides an approximation of the binomial distribution when n is large and p is small and np is moderate magnitude.
- Assumptions:
  (1) The number of occurrences in non-overlapping intervals are independent.
  (2)Chance of 2 or more occurrences can be assumed to be 0
  (3) Average rate of occurrences=m per unit time is a constant.
- The probability distribution of X is

$$P(X = x) = \frac{e^{-m}\ m^x}{x!},\ x = 0, 1, 2, ...,$$

Cont./ Poisson Distribution:

- The mean $\mu = E(X) = m$ and $\sigma^2 = m$.
- Let us consider the following example. $X =$ number of accidents on a highway per day. Assume X is Poisson with $m = 2$. Then $\mu = E(X) = 2$ and Std.=$\sqrt{2}$.

$$\begin{aligned} P(1 < X \leq 4) &= P(X \leq 4) - P(X \leq 1) \\ &= 0.947 - 0.406 = 0.541 \end{aligned}$$

$$\begin{aligned} P(X = 2) &= P(X \leq 2) - P(X \leq 1) \\ &= 0.677 - 0.406 = 0.271 \end{aligned}$$

If Y is the number of accidents per 2 days, then $m = 2(2) = 4$. Therefore

$$P(Y = 2) = 0.238 - 0.092 = 0.146.$$

Poisson Approximation to Binomial:

90 people are taken under medical test with P(getting a cold)=0.1. Find an approximate value of the probability that at least 10 persons have a cold

X=number persons with a cold $Bin(90, 0.1)$.

$$\begin{aligned} P_B(X \geq 10) &\approx P_P(X \geq 10) \\ &= 1 - P(X \leq 9) \\ &= 1 - 0.587 \\ &= 0.413. \end{aligned}$$