Normal Distribution

Normal Probabilities:

- X is continuous r.v. with $f(x)$ is the probability density function (pdf) and area under the curve of $f(x)$ is 1. This pdf $f(x)$ is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \; e^{-\frac{1}{\sigma^2}(x-\mu)^2}, \; -\infty < x < \infty.$$

- $P(a \leq X \leq b)$ = area under $f(x)$ between a and b.
- $P(X = c) = 0$
- $P(a \leq X \leq b) = P(a < X \leq b) = P(a < X < b)$
- $P(a < X < b)$= Area to the left of b- Area to the left of a.
- $P(X > b) = 1 - P(X \leq b)$.
- $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. There is one table gives the area under some values. Therefore,

$$P(a < Z < b) = T(b) - T(a),$$

$T(c)$ is the value of the table under c.

Examples

(a) $P(Z < 1.25) = T(1.25) = 0.8944$

(b)$P(0 < Z < 1.25) = T(1.25) = 0.8944 - 0.5 = 0.3944$

(c)$P(Z > -1.25) = 1 - T(-1.25) = 1 - 0.1056 = 0.8944$

(d)$P(-1.25 < Z < 1.25) = T(1.25) - T(-1.25) = 0.8944 - 0.1056 = 0.7888$

(e) $P(0.5 < Z < 1.25) = T(1.25) - T(0.5) = 0.8944 - 0.6915 = 0.2029$

(f) $P(-0.5 < Z < 1.25) = T(1.25) - T(-0.5) = 0.8944 - 0.3085 = 0.5859$

(g) Locate the value of c satisfying $P(Z > c) = 0.05$, then c is 95th percentile. That is $c = 1.64$.

(h) Find the value of c satisfying $P(-c < Z < c) = 0.90$, then c is 95th percentile. That is $c = 1.64$.

Empirical rule

Given the distribution of measurements is approximately bell-shaped then

(1) $\approx$ 68% of the measurements will be within $\mu \pm \sigma$.

(2) $\approx$ 95% of the measurements will be within $\mu \pm 2\,\sigma$.

(3) $\approx$ 99% of the measurements will be within $\mu \pm 3\sigma$.

Check:

$$\begin{aligned} P(\mu - \sigma < X < \mu + \sigma) &= P(-1 < Z < 1) \\ &= 0.8413 - 0.1587 \\ &= 0.6826. \end{aligned}$$

$$\begin{aligned} P(\mu - 2\sigma < X < \mu + 2\sigma) &= P(-2 < Z < 2) \\ &= 0.9772 - 0.0228 \\ &= 0.9544. \end{aligned}$$

Example

Assume that your scores in Math.131 are normally distributed with mean $\mu = 65$ and $Std. = 5$. Find

(1) $P(60 < X < 70)$

$$\begin{aligned} P(60 < X < 70) &= P(\frac{60-65}{5} < Z < \frac{70-65}{5}) \\ &= P(-1 < Z < 1) \\ &= T(1) - T(-1) = 0.6826. \end{aligned}$$

(2) $P(X < 75)$

$$\begin{aligned} P(X > 75) &= P(Z > \frac{75-65}{5}) \\ &= P(Z > 2) \\ &= 1 - 0.9772 = 0.0228. \end{aligned}$$

Cont./ Example

(3) If 10% of students will get grade A in this course, what is the minimum score to get an A.
$P(Z \leq z) = 0.9 \rightarrow z = 1.28$. This is a standardized value which is equivalent to $X = 65 + 5(1.28) = 71.40$.

Example: The grades of section 1: $X \sim N(50, 4)$ and the grades of section 2: $Y \sim N(55, 5)$. One student from each section is taken, what is the probability that the score of section 1 student is greater than that of section 2.

$$\begin{aligned} P(X > Y) &= P(X - Y > 0) \\ &= P(\frac{X - Y - 5}{3} > \frac{5}{3}) \\ &= P(Z > 1.67) \\ &= 0.0485. \end{aligned}$$

Note that if $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$, then

(1) $Y = a + bX \sim N(a + b\mu, b^2\sigma_1^2)$. (2) $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Normal Approximation to Binomial

- If $X \sim B(n,p)$, then $\mu = np$ and $\sigma = \sqrt{npq}$.
- For n large and np and npq are moderate $[np\&nq > 15]$, the normal distribution can be used to approximate the binomial distribution.
- $$\begin{aligned} P_B(a < X < b) &\approx P(a - 0.5 \le X \le b + 0.5) \\ &= P(\frac{a - 0.5 - np}{\sqrt{npq}} > \frac{b + 0.5 - np}{\sqrt{npq}}) \end{aligned}$$
- Consider a sample of size 50 items taken for a special test. Given that P(an item is defective)=0.6. We decide to reject the product if X=number of defective items ≥ 30. Use the normal approximation to compute the P(product is rejected).

$$\begin{aligned} P_B(X \geq 30) &\approx P_N(X \geq 29.5) \\ &= P(Z \geq \frac{30 - 0.5 - 30}{3.46}) \\ &= P(Z \geq -0.144) = 1 - 0.4434 = 0.5557 \end{aligned}$$

Cont./Example

- Find $P_B(25 \leq X < 35)$. Now

$$\begin{aligned} P_B(25 \leq X < 35) &\approx P_N(24.5 \leq X \leq 34.5) \\ &= P(\frac{24.5-30}{3.46} \leq Z \leq \frac{34.5-30}{3.46}) \\ &= P(-1.59 \leq Z \leq 1.3) \\ &= 0.9032 - 0.0559 = 0.8473 \end{aligned}$$