Concepts of Statistical Inference

Testing Hypotheses

- **Definition:** A statistical hypothesis is a statement about the population. The hypothesis is either null hypothesis (H_0) or alternative hypothesis (H_1). Based on the data, we decide one of these hypotheses is true. That is,
  (a) Reject H_0: the assertion is false
  (b) Accept H_0: the assertion is true

- One of the decisions can be reached by using information from the sample observations. The process of testing hypotheses is similar to the procedures used in a court trial. The prosecution collects and presents all available evidence in an attempt to contradict "not guilty" hypothesis.

- Let H_0: not guilty vs. H_1: guilty

	H_0 is true	H_0 is false
Reject H_0	Type I error	Oki
Accept H_0	Oki	Type II error

Errors

- Type I error: reject H_0 when H_0 true.
- Type II error: accept H_0 when H_1 is true.
- $\alpha = P$(Type I error)=P(reject $H_0|H_0$).
- $\beta = P$(Type II error)=P(accept $H_0|H_1$).
- The power function $\gamma(\theta) =$P(reject $H_0|\theta$). The power of the test P(reject $H_0|H_1$) indicate the strength and goodness of the test.

  Example (Quality Control): Suppose that a production process has been in control (production process is stable) for some time and the proportion of defectives has been 0.05. For monitoring the process, the staff decides to consider the process "out of control" if 2 or more defectives are found in a sample of 15 items.

Cont./Example

(a)The power function at $p = 0.05, 0.1, 0.2, 0.3, 0.4$. The hypotheses can be written as $H_0 : p = 0.05$ versus $H_1 : p > 0.05$.

p	0.05	0.1	0.2	0.3	0.4
	0.171	0.451	0.833	0.965	0.995

(b) Compute α

$$\alpha = P(X \geq 2 | p = 0.05) = 1 - T(1) = 0.171$$

Note that $X \sim Bin(15, p)$.
(c) Compute β at $p = 0.1$.

$$\beta = P(X < 2 | p = 0.1) = T(1) = 0.549$$

(d) Determine the rejection region so that $\alpha \approx 0.04$.

$$\alpha = 0.04 = P(X \geq c) = 1 - P(X \leq c - 1)$$

Cont./Example

$\Rightarrow P(X \leq c-1) = 0.96$. Then $c-1=2$ or $c=3$. That is, reject H_0 if $X \geq 3$.

Example: Assume the age of electrical bulbs is normally distributed with Std. $\sigma = 400$. The manufacturer claims that the mean age will last more than 5000 hours. To test

$$H_0 : \mu = 5000 \ vs. \ \ H_1 : \mu > 5000.$$

we reject H_0 if $\overline{X} > 5075$
(a) Compute the significance level α

$$\begin{aligned} \alpha = P(reject H_0|H_0) &= P(\overline{X} > 5075|\mu = 5000) \\ &= P(Z > 1.88) = 1 - 0.9699 = 0.0301 \end{aligned}$$

(b) Compute β at $\mu = 5045$.

$$\begin{aligned} \beta = P(accept H_0|H_1) &= P(\overline{X} \leq 5075|\mu = 5045) \\ &= P(Z \leq 0.75) = 0.7734 \end{aligned}$$

Cont./ Example

(c) Determine the rejection region for which $\alpha = 0.10$

$$\alpha = 0.10 = P(\overline{X} > c|\mu = 5000)$$
$$0.10 = P(Z > \frac{c-5000}{40})$$
$$\Rightarrow \frac{c-5000}{40} = 1.28$$
$$c = 5051.2$$

That is, reject H_0 if $\overline{X} > 5051.2$.

p-value

- The p-value is the smallest significance level at which H_0 is rejected based on the observed sample. Using the p-value, we reject H_0 if p-value<α and we accept H_0 if p-value $\geq \alpha$.

Example: Given $n = 100, \overline{X} = 80, \sigma = 20$. Determine the p-value for testing $H_0 : \mu = 75$ vs. $H_1 : \mu > 75$.

$$\begin{aligned} p - value &= = P(\overline{X} > 80) \\ &= P(Z > \frac{80 - 75}{20/10}) \\ &= P(Z > 2.5) = 0.0062. \end{aligned}$$

For testing $H_0 : \mu = 75$ vs. $H_1 : \mu \neq 75$.

$$\begin{aligned} p - value &= = 2P(\overline{X} > 80) \\ &= 2P(Z > \frac{80 - 75}{20/10}) \\ &= 2P(Z > 2.5) = 2(0.0062) = 0.0124. \end{aligned}$$