Hypotheses Testing

$X \sim P_\theta \,,\ \mathcal{P}_\theta = \{P_\theta : \theta \in \Theta\}$

$H_0 : \theta \in \Theta_0$ vs. $H_1 : \theta \in \Theta_1$ such that $\Theta = \Theta_0 \cup \Theta_1$
with $\Theta_0 \cap \Theta_1 = \phi$ "empty set"

Def.: A non-randomized test of H_0 versus H_1 can be specified by a critical region S that lead to accept H_1 if $X \in S$ and accept H_0 if $X \notin S$. That is,

$$\Omega = \underset{\substack{\uparrow \\ \text{rejection} \\ \text{region}}}{S} \cup \underset{\substack{\uparrow \\ \text{acceptance} \\ \text{region}}}{\bar{S}}$$

Type I error: reject H_0 when it is true
Type II error: accept H_0 " " " false

Problem: Partition Ω so as to min. Type II error subject to constraint Type I error $= P(X \in S) \le \alpha$, $\forall\, \theta \in \Theta_0$ with α being the significance level $(0 < \alpha < 1)$.

Def.: $\alpha = \sup_{\theta \in \Theta_0} P_\theta(X \in S)$ is the size of the test

Def.: The function $\beta : \Theta \longrightarrow [0,1]$ defined by $\beta(\theta) = P_\theta(X \in S)$ is called power function

- Ideally, we would want $\beta(\theta) = 0$ for $\theta \in \Theta_0$ and $\beta(\theta) = 1$ for $\theta \in \Theta_1$, but it is impossible in practice.

Note: For technical reason, it is appropriate to adopt "randomized test" to choose better between H_0 and H_1.

Def.: A randomized test is determined by a critical function
$\phi : \Omega \longrightarrow [0,1]$, where
$\phi(X) = P(\text{rej. } H_0 \text{ given } X)$

Problem [for randomized test]: For each $\theta \in \Theta$, we need to maximize

(1) $\beta(\theta) = \beta_\phi(\theta) = E_\theta\, \phi(X)$ subject to the level constraint

(2) $E_\theta[\phi(X)] \le \alpha, \ \forall\, \theta \in \Theta_0$.

Special Case: If Θ_0 and Θ_1, each consists of a single prob. dist., then a test ϕ is most powerful of size α (MP_α) if it maximizes (1) subject to (2) among all randomized tests.

Example: $\Omega = \{1,2,3,4\}$, $\Theta_0 = \{P_0\}$; $\Theta_1 = \{P_1\}$

X	1	2	3	4
$P_0(x)$	0.9	0.05	0.03	0.02
$P_1(x)$	0.1	0.5	0.40	0

- What is the MP level 0.05 nonrandomized test?
  $S = \{2\}$, $P_{H_1}(X=2) = 0.5$
- What is the $MP_{.05}$ randomized test?
  Let us consider
  $$\phi(X) = \begin{cases} 0 & \text{if } X = 1, 4 \\ \gamma_1 & \text{if } X = 2 \\ \gamma_2 & \text{if } X = 3 \end{cases}$$

$E_0\phi(X) = 0.05\,\gamma_1 + 0.03\,\gamma_2 \Rightarrow \alpha = 0.05\,\gamma_1 + 0.03\,\gamma_2 = 0.05$

$\Rightarrow \boxed{\gamma_1 = 1 - \frac{3}{5}\gamma_2}$ ---(*)

Now, we maximize $E_{H_1}\phi(X) = 0.5\gamma_1 + 0.4\gamma_2 = 0.5 - 0.3\gamma_2 + 0.4\gamma_2$

$= 0.5 + 0.1\,\gamma_2$ by using (*)

$\gamma_2 = 1$ maximizes $E_{H_1}\phi(X) \Rightarrow \gamma_1 = 1 - \frac{3}{5} = 0.4$

$$\Rightarrow \quad \phi(X) = \begin{cases} 0 & \text{if } X = 1, 4 \\ 0.4 & \text{if } X = 2 \\ 1 & \text{if } X = 3 \end{cases}$$

Check: (1) size of $\phi(X) = E_0\phi(X) = 0.4 \times .05 + 1 \times .03 = 0.05$ ✓

(2) power of $\phi(X) = E_1\phi(X) = 0.4 \times .5 + 1 \times .4 = 0.60$

> 0.5 for S

Simple vs. Simple Testing

$H_0 : X \sim P_0$ vs. $H_1 : X \sim P_1$ [2 dist'ns for the data X]

- $\alpha = E_0\phi(X) = \int \phi(x)\, P_0(x)\, dx$
- $E_1\phi(X) = \int \phi(x)\, P_1(x)\, dx$

Let us consider the constraint maximization problem with Lagrange Multipliers

Proposition: Suppose $k \geq 0$, ϕ^* maximizes $E_1\phi - kE_0\phi$ among all critical functions and $E_0\phi^* = \alpha$. Then ϕ^* maximizes $E_1\phi$ over all ϕ with level at most α.

Proof: ~~Suppose~~ Suppose ϕ has level at most α,

$$E_0 \phi \le \alpha$$

$$\begin{aligned} E_1 \phi &= E_1 \phi - k E_0 \phi + k E_0 \phi \\ &\le E_1 \phi - k E_0 \phi + k \alpha \\ &\le E \phi^* - k E_0 \phi^* + k \alpha \\ &= E_1 \phi^* \end{aligned}$$

Maximization $E_1 \phi - k E_0 \phi$ is an easy task since

$$\begin{aligned} E_1 \phi - k E_0 \phi &= \int [p_1(x) - k p_0(x)] \phi(x)\, dx \\ &= \int_{p_1(x) > k p_0(x)} |p_1(x) - k p_0(x)| \phi(x)\, dx \\ &\quad - \int_{p_1(x) < k p_0(x)} |p_1(x) - k p_0(x)| \phi(x)\, dx \end{aligned}$$

$\Rightarrow$ Any test ϕ^* maximizing this expression must have

$$\phi^*(x) = 1 \quad \text{when } p_1(x) > k p_0(x)$$
$$= 0 \quad \text{when } p_1(x) < k p_0(x)$$

This means that the test is based on the likelihood ratio $L(x) = p_1(x)/p_0(x)$ with $\phi^*(x) = 1$ if $L(x) > k$ and $= 0$ if $L(x) < k$.

When $L(x) = k$, $\phi(x)$ can take any value in $[0,1]$.

Neyman-Pearson Lemma:

Let P_0 and P_1 have probability densities p_0 & p_1 with respect to some measure μ on Ω

(1) Existence of a LR (likelihood ratio) test of size α.
For testing $H_0: X \sim P_0$ vs. $H_1: X \sim P_1$, $\exists$ at least $\phi(X)$ of the form

$$\phi(x) = \begin{cases} 1 & \text{if } p_1(x) > k\, p_0(x) \text{ with } k > 0 \\ 0 & \text{if } p_1(x) < k\, p_0(x) \end{cases} \qquad (**)$$

[Any test of the form (**) is called LR test]

(2) ϕ LR test of size $\alpha \Rightarrow MP_\alpha$

(3) ϕ MP_α test $\Rightarrow \phi$ LR of size α

Def.: A (randomized) test ϕ is $UMP_\alpha \iff E_\theta \phi \le \alpha, \forall \theta \in \Theta_0$ and for any other α test ϕ', $E_\theta[\phi] \ge E_\theta[\phi']$, $\forall \theta \in \Theta_1$.

New Setting: $\Theta = \mathbb{R}$, $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$

Def.: A family of densities $\{P_\theta \mid \theta \in \Theta\}$ is said to be a monotone likelihood ratio (MLR) family in a non-constant statistic $T(X)$ if

(1) $\theta \ne \theta' \Rightarrow P_\theta \ne P_{\theta'}$ where $dP_\theta(x) = p_\theta(x)\,dx$
[The dist. should be identifiable]

(2) $\exists\, T: \Omega \longrightarrow \mathbb{R} \ni \theta < \theta' \Rightarrow \dfrac{p_{\theta'}(x)}{p_\theta(x)} = g_{\theta,\theta'}(T(x))$ for some $g_{\theta,\theta'}$, is non-decreasing.

Examples

① Suppose X is absolutely continuous with density

$$p_\theta(x) = \theta e^{-\theta x}, \quad x > 0$$

Want a LR test ϕ for $H_0: \theta = 1$ vs. $H_1: \theta = \theta_1$ $(\theta_1 > 1)$

$$\frac{p_1(x)}{p_0(x)} = \frac{\theta_1 e^{-\theta_1 x}}{e^{-x}} > k \iff \theta_1 e^{-(\theta_1 - 1)x} > k \iff X < k'$$

$$\Rightarrow \phi(x) = \begin{cases} 1 & \text{if } X < k' \\ 0 & \text{if } X > k' \end{cases}$$

When $X = k'$, $\phi(x)$ can take any value in $[0,1]$ ~~but the~~ but this will not affect any power calculation since $P(X = k') = 0$.

The level of this test is

$$\alpha = P_{\theta=1}(X < k') = \int_0^{k'} e^{-x}\,dx = 1 - e^{-k'} \Rightarrow k' = -\log(1-\alpha).$$

The simplified LR test of size α is

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } X < -\log(1-\alpha) \\ 0 & \text{if } X > -\log(1-\alpha) \end{cases}$$

Note that $E_{\theta_1}\phi(X) \le E_{\theta_1}\phi_\alpha(X)$ for all $\theta_1 > 1$.

② Show that $N(\theta, 1)$ pdf is MLR in $T(X) = X$

For $\theta < \theta'$,

$$\frac{p_{\theta'}(x)}{p_\theta(x)} = e^{-\frac{1}{2}(x-\theta')^2 + \frac{1}{2}(x-\theta)^2} = e^{x(\theta'-\theta) + \frac{1}{2}(\theta^2 - \theta'^2)} \nearrow \text{ in } X$$

③ Is $p_\theta(x) = I_{(\theta, \theta+1)}(x)$, $\theta \in \mathbb{R} = \Theta$ MLR? Why?

For $\theta < \theta'$,

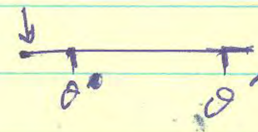
$$\frac{p_{\theta'}(x)}{p_\theta(x)} = \frac{I\{\theta' < x < \theta'+1\}}{I\{\theta < x < \theta+1\}} = \begin{cases} 0/0 \equiv c & \text{for } x < \theta \\ 0/1 & \text{for } \theta < x < \min(\theta', \theta+1) \\ 1 \text{ or } c & \text{depending on } \theta+1 > \theta' \text{ or } \theta < \theta'+1 \\ 1/0 & \text{for } \max(\theta+1, \theta') < x < \theta'+1 \\ 0/0 & \text{for } \theta'+1 < x \end{cases}$$

$\Rightarrow$ Clearly not increasing $\Rightarrow$ Not MLR

4 Let $X_1, X_2, \ldots, X_n \overset{r.s.}{\sim} U(0,\theta)$, $\theta>0$. The joint pdf of $X_1,\ldots,X_n$ is

$$p_\theta(\underset{\sim}{x}) = \frac{1}{\theta^n}, \; 0< x_{(n)}<\theta$$

Let $\theta<\theta'$, $$\frac{p_{\theta'}(\underset{\sim}{x})}{p_\theta(\underset{\sim}{x})} = \left(\frac{\theta}{\theta'}\right)^n \frac{I\{0<x_{(n)}<\theta'\}}{I\{0<x_{(n)}<\theta\}}$$



$$= \begin{cases} \left(\frac{\theta}{\theta'}\right)^n & \text{if } 0<x_{(n)}\le\theta \\ \infty & \text{if } \theta\le x_{(n)}\le\theta' \end{cases}$$

$\Rightarrow$ It is non-decreasing function in $T(\underset{\sim}{x}) = X_{(n)}$. The family of uniform densities on $[0,\theta]$ has an MLR in $X_{(n)}$.

5 Let $X \sim$ Cauchy dist. $C(1,\theta)$ with

$$p_\theta(x) = \frac{1}{\pi[1+(x-\theta)^2]}, \; -\infty<x<\infty.$$

For $\theta'>\theta$, $$\frac{p_{\theta'}(x)}{p_\theta(x)} = \frac{1+(x-\theta)^2}{1+(x-\theta')^2} \longrightarrow 1 \text{ as } x\longrightarrow\pm\infty$$

$\Rightarrow C(1,\theta)$ does not have MLR.

6 If $p_\theta(x) = e^{\eta(\theta)T(x)-B(\theta)} \cdot h(x)$ exp. family. Then $\{P_\theta \mid \theta\in\Theta\}$ is MLR in $T(x)$ if $\eta(\theta)$ is strictly increasing or it is MLR in $(-T(x))$ if $\eta(\theta)$ is strictly decreasing

To see that, for $\theta'>\theta$

$$\frac{p_{\theta'}(x)}{p_\theta(x)} = e^{[\eta(\theta')-\eta(\theta)]T(x)+(B(\theta)-B(\theta'))}$$

$\nearrow$ if $\eta(\theta)$ is increasing

$\searrow$ if $\eta(\theta)$ is decreasing.

Theorem: Suppose $\{P_\theta\}$ is MLR family in $T(X)$. Then

(i) For $0 \le \alpha \le 1$, $\exists$ a UMP$_\alpha$ test for $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$ of the form
$$\phi(X) = \begin{cases} 1 & \text{if } T(X) > c \\ \gamma & \text{if } T(X) = c \\ 0 & \text{if } T(X) < c \end{cases}$$

(ii) $\beta(\theta) = E_\theta[\phi(X)]$ is strictly increasing over $\{\theta : 0 < \beta(\theta) < 1\}$

(iii) ϕ is UMP$_{\alpha'}$ for testing $H_0: \theta \le \theta'$ vs. $H_1: \theta > \theta'$ where $\alpha' = \beta(\theta')$

(iv) $\forall\, \theta < \theta_0$, ϕ has $\beta(\theta) = \inf E_\theta[\phi^*]$

Examples

(1) Let X have the pdf
$$P_M(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, \quad x = 0, 1, \ldots, M \quad [\text{Hypergeometric dist}]$$
$$\frac{P_{M+1}(x)}{P_M(x)} = \frac{M+1}{N-M} \cdot \frac{N-M-n+x}{M+1-x} \nearrow \text{ in } x$$

The UMP test of $H_0: M \le M_0$ vs. $M > M_0$ is
$$\phi(X) = \begin{cases} 1 & \text{if } X > k \\ \gamma & \text{if } X = k \\ 0 & \text{if } X < k \end{cases} \quad \text{where } E_{M_0}\phi(X) = \alpha.$$

Theorem: For the one-parameter exponential family, $\exists$ a UMP test of $H_0: \theta \le \theta_1$, or $\theta \ge \theta_2$ $(\theta_1 < \theta_2)$ against $H_1: \theta_1 < \theta < \theta_2$ that is of the form:
$$\phi(X) = \begin{cases} 1 & \text{if } c_1 < T_1(X) < c_2 \\ \gamma_i & \text{if } T(X) = c_i,\ i = 1, 2 \ (c_1 < c_2) \\ 0 & \text{if } T(X) < c_1 \text{ or } T(X) > c_2 \end{cases} \quad \text{where } \boxed{E_{\theta_1}\phi(X) = E_{\theta_2}\phi(X) = \alpha}$$

Example: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, 1)$. To test $H_0: \mu \le \mu_0$ or $\mu \ge \mu_1$ ($\mu_1 > \mu_0$) against $H_1: \mu_0 < \mu < \mu_1$, the UMP test

$$\phi(\underset{\sim}{X}) = \begin{cases} 1 & \text{if } c_1 < \sum^n X_i < c_2 \\ \gamma_i & \text{if } \sum X_i = c_1 \text{ or } c_2 \\ 0 & \text{if } \sum X_i < c_1 \text{ or } \sum X_i > c_2 \end{cases}$$

where

$$\alpha = P_{\mu_0}(c_1 < \sum X_i < c_2) = P_{\mu_1}(c_1 < \sum X_i < c_2)$$

$$= P\left(\frac{c_1 - n\mu_0}{\sqrt{n}} < Z < \frac{c_2 - n\mu_0}{\sqrt{n}}\right) = P\left(\frac{c_1 - n\mu_1}{\sqrt{n}} < Z < \frac{c_2 - n\mu_1}{\sqrt{n}}\right)$$

where $Z \sim N(0,1)$. Equivalently,

$$\Phi\left(\frac{c_2 - n\mu_0}{\sqrt{n}}\right) - \Phi\left(\frac{c_1 - n\mu_0}{\sqrt{n}}\right) = \alpha$$

$$\Phi\left(\frac{c_2 - n\mu_1}{\sqrt{n}}\right) - \Phi\left(\frac{c_2 - n\mu_1}{\sqrt{n}}\right) = \alpha$$

where Φ is the cdf of Z.

Example: Let $X_1, \ldots, X_n \overset{iid}{\sim} N(0, \sigma^2)$. This family has MLR in $\sum X_i^2$. Can we obtain UMP test for $H_0: \sigma^2 = \sigma_0^2$ vs. $H_1: \sigma^2 \ne \sigma_0^2$.

For $H_0: \sigma^2 = \sigma_0^2$ vs. $H_1: \sigma^2 > \sigma_0^2$, $\phi_1(X) = \begin{cases} 1 & \text{if } \sum X_i^2 > c_1 \\ 0 & \text{e.w.} \end{cases}$ UMP

For $H_0: \sigma^2 = \sigma_0^2$ vs. $H_1: \sigma^2 < \sigma_0^2$, $\phi_2(X) = \begin{cases} 1 & \text{if } \sum X_i^2 < c_2 \\ 0 & \text{e.w.} \end{cases}$ UMP

$\Rightarrow c_1 = \sigma_0^2 \chi_n^2(\alpha)$, $c_2 = \sigma_0^2 \chi_n^2(1-\alpha)$.

Neither ϕ_1 nor ϕ_2 is UMP for $H_0: \sigma^2 = \sigma_0^2$ vs. $H_1: \sigma^2 \ne \sigma_0^2$

Unbiased Tests

There are many practical situations when the UMP tests do not exist. These situations comprise, among others, those of testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$. That is, we may have a UMP unbiased test.

Def. The test ϕ of H_0 against H_1 is called unbiased if

$$\alpha = \sup_{\theta \in \Theta_0} \beta_\phi(\theta) \leq \inf_{\theta \in \Theta_1} \beta_\phi(\theta)$$

Equivalently, $\beta_\phi(\theta) \leq \alpha$ for $\theta \in \Theta_0$
and $\beta_\phi(\theta) \geq \alpha$ for $\theta \in \Theta_1$

Def. Let U_α be the class of all unbiased size α tests of H_0. If there exists a test $\phi \in U_\alpha$ that has maximum power at each $\theta \in \Theta_1$, we call ϕ a UMP unbiased size α test.

Note: In particular, if the null hypothesis is simple, then the power function of an unbiased test reaches its minimum at $\theta = \theta_0$.

Example: $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, 1)$. Wish to test $H_0: \mu \leq 0$ vs. $H_1: \mu > 0$. It was shown that $N(\mu,1)$ has MLR in $\sum X_i$. The UMP test is to reject H_0 if $\sum^n X_i > c$, where c is determined by

$$\alpha = P_0\left(\sum X_i > c\right) = P\left(\bar{X} > \frac{c}{\sqrt{n}}\right) = 1 - \Phi\left(\frac{c}{\sqrt{n}}\right)$$

$$\Rightarrow \frac{c}{\sqrt{n}} = z_{1-\alpha} \Rightarrow c = z_{1-\alpha}\sqrt{n}$$

$$\Rightarrow \phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum x_i > \sqrt{n}\, z_{1-\alpha} \\ 0 & \text{e.w.} \end{cases}$$

$$\beta_\phi(\mu) = P_\mu\left(\sum X_i > \sqrt{n}\, z_{1-\alpha}\right) = 1 - \Phi\left(\frac{\sqrt{n}\, z_{1-\alpha} - n\mu}{\sqrt{n}}\right) = 1 - \Phi\left(z_{1-\alpha} - \sqrt{n}\mu\right)$$

$\Rightarrow \beta_\phi(\mu) \leq \alpha$ for H_0 and $\beta_\phi(\mu) > \alpha$ for H_1.

Here, we consider a UMP unbiased test since there is no UMP test.

Example: Consider the test for $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$ in case of obs's $X_1, X_2, \dots, X_n$ being a r.s. from $N(\mu, \sigma^2)$ with known σ^2. It is easy to show $\nexists$ UMP test. Let us take $T(\underset{\sim}{X}) = \bar{X} - \mu_0$ as the test statistic and reject H_0 if $T(\underset{\sim}{X}) < -k'$ or $T(\underset{\sim}{X}) > k''$ for some k' and k'', such that

$$
\begin{aligned}
1-\alpha &= P_{\mu_0}\left(-k' < T(\underset{\sim}{X}) < k''\right) \\
&= P_{\mu_1}\left(\frac{-k'\sqrt{n}}{\sigma} < Z < \frac{k''\sqrt{n}}{\sigma}\right) \\
&= \Phi\left(\frac{k''\sqrt{n}}{\sigma}\right) - \Phi\left(\frac{-k'\sqrt{n}}{\sigma}\right) \quad \text{where } \Phi \text{ is the cdf of } N(0,1).
\end{aligned}
$$

The power function of $T(\underset{\sim}{X})$ is

$$
\begin{aligned}
\beta(\mu) &= 1 - P(\mu_0 - k' < \bar{X} < \mu_0 + k'') \\
&= 1 - \Phi\left(\frac{\mu_0 - \mu + k''}{\sigma}\sqrt{n}\right) + \Phi\left(\frac{\mu_0 - \mu - k'}{\sigma}\sqrt{n}\right)
\end{aligned}
$$

continuous diff. of μ

$\phi(\underset{\sim}{x})$ is unbiased if $\left.\frac{d}{d\mu}\beta(\mu)\right|_{\mu=\mu_0} = 0$

$$\frac{d\beta(\mu)}{d\mu} = \varphi\left(\frac{\mu_0 - \mu + k''}{\sigma}\sqrt{n}\right)\frac{\sqrt{n}}{\sigma} - \varphi\left(\frac{\mu_0 - \mu - k'}{\sigma}\sqrt{n}\right)\frac{\sqrt{n}}{\sigma}$$

$$\Rightarrow \left.\frac{d\beta(\mu)}{d\mu}\right|_{\mu=\mu_0} = \left[\varphi\left(\frac{k''}{\sigma}\sqrt{n}\right) - \varphi\left(\frac{-k'}{\sigma}\sqrt{n}\right)\right]\frac{\sqrt{n}}{\sigma} = 0$$

$$\Leftrightarrow k' = k'' = k$$

$$\Leftrightarrow \cancel{k = z_{1-\alpha/2}\frac{\sqrt{n}}{\sigma}} \quad \frac{\sqrt{n}}{\sigma}k = z_{1-\alpha/2}$$

$$k = \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}$$

Invariant Test

Def.: A group $\mathcal{G}$ of transformations on the space of $\underset{\sim}{X}$ leaves a hypothesis testing problem invariant if $\mathcal{G}$ leaves both $\{P_\theta : \theta \in \Theta_0\}$ and $\{P_\theta : \theta \in \Theta_1\}$ invariant.

Def.: We say ϕ is invariant under $\mathcal{G}$ if $\phi(g\underset{\sim}{x}) = \phi(\underset{\sim}{x})$ $\forall\, \underset{\sim}{x}, g \in \mathcal{G}$.

Def.: Let $\mathcal{G}$ be a group of transformations on the space of $\underset{\sim}{X}$. $T(\underset{\sim}{X})$ is maximal invariant under $\mathcal{G}$ if

(a) T is ~~maximal~~ invariant

(b) T is maximal, that is, $T(\underset{\sim}{x}_1) = T(\underset{\sim}{x}_2) \Rightarrow \underset{\sim}{x}_1 = g(\underset{\sim}{x}_2)$ for some $g \in \mathcal{G}$

Examples:

① Let $\mathcal{G}$ be the group of translations $g_c(\underset{\sim}{x}) = (x_1 + c, \dots, x_n + c)$, where $-\infty < c < \infty$. Consider $T(\underset{\sim}{x}) = (x_1 - x_n, \dots, x_{n-1} - x_n)$,

(i) $T(g_c \underset{\sim}{x}) = (x_1 - x_n, \dots, x_{n-1} - x_n) = T(\underset{\sim}{x})$

(ii) If $T(\underset{\sim}{x}) = T(\underset{\sim}{x}') \Rightarrow x_i - x_n = x_i' - x_n'$, $i = 1, \dots, n-1$

$$\Rightarrow x_i - x_i' = x_n - x_n', \quad i = 1, \dots, n-1$$
$$= c$$

$$\Rightarrow x_i' = x_i - c \Rightarrow g_c(\underset{\sim}{x}') = (x_1' + c, \dots, x_n' + c)$$
$$= (x_1 - c + c, \dots, x_n - c + c) = \underset{\sim}{x}$$

$\Rightarrow T$ is maximal invariant.

② Same set-up as before but $g_c(\underset{\sim}{x}) = (cx_1, \dots, cx_n)$, $c > 0$

Then $T(\underset{\sim}{x}) = \left(\dfrac{x_1}{\sqrt{\sum x_i^2}}, \dots, \dfrac{x_n}{\sqrt{\sum x_i^2}} \right)$

(i) $T(g_c(\underset{\sim}{x})) = T(cx_1, \dots, cx_n) = T(\underset{\sim}{x})$

(ii) Let $T(\underset{\sim}{x}) = T(\underset{\sim}{x}') \Rightarrow \dfrac{x_i}{\sqrt{\sum x_i^2}} = \dfrac{x_i'}{\sqrt{\sum x_i'^2}}$

$\Rightarrow \; x_i' = \left(\frac{z'}{z}\right) x_i = c x_i$, where $c = \left(\frac{\sum x_i'^2}{\sum x_i^2}\right)^{1/2}$.

$\Rightarrow T$ is maximal invariant.

Also, if we consider $g(\underset{\sim}{x}) = (a x_1 + b, \ldots, a x_n + b)$, $a > 0$, $b \in (-\infty, \infty)$, then

$$T(\underset{\sim}{x}) = \left(\frac{x_1 - \bar{x}}{S}, \ldots, \frac{x_n - \bar{x}}{S}\right), \text{ where } S^2 = \frac{\sum (x_i - \bar{x})^2}{n} \text{ is}$$

maximal invariant.

__Def.:__ Let I_α denote the class of all invariant size α tests of $H_0: \theta \in \Theta_0$ versus $H_1: \theta \in \Theta_1$. If there is a UMP in I_α, we call it UMP invariant test of H_0 against H_1.

__Theorem:__ Let $T(\underset{\sim}{x})$ be a maximal invariant with respect to $\mathcal{G}$. Then ϕ is invariant under $\mathcal{G}$ iff ϕ is a function of T.

__Proof:__ $\Rightarrow$ Need to show ϕ is a function of T. That is if

$$T(\underset{\sim}{x}_1) = T(\underset{\sim}{x}_2) \Rightarrow \phi(\underset{\sim}{x}_1) = \phi(\underset{\sim}{x}_2)$$

$$T(\underset{\sim}{x}_1) = T(\underset{\sim}{x}_2) \Rightarrow \exists\, g \in \mathcal{G} \ni \underset{\sim}{x}_1 = g\underset{\sim}{x}_2 \Rightarrow \phi(\underset{\sim}{x}_1) = \phi(g\underset{\sim}{x}_2) = \phi(\underset{\sim}{x}_2)$$

$\Leftarrow$ Let ϕ be a function of $T \Rightarrow \phi(\underset{\sim}{x}) = h(T(\underset{\sim}{x}))$

$$\Rightarrow \phi(g\underset{\sim}{x}) = h(T(g\underset{\sim}{x})) = h(T(\underset{\sim}{x})) = \phi(\underset{\sim}{x})$$

$\Rightarrow \phi$ is invariant.

__Example:__ Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where both μ and σ^2 are unknown. We wish to test $H_0: \sigma \geq \sigma_0$, $-\infty < \mu < \infty$

$H_1: \sigma < \sigma_0$, $-\infty < \mu < \infty$

— The family $\{N(\mu, \sigma^2)\}$ remains invariant under translations

$x_i' = x_i + c$, $-\infty < c < \infty$.

- The hypothesis testing problem remains invariant under the group of translations. That is, $\{N(\mu, \sigma^2): \sigma^2 \geq \sigma_0^2\}$ & $\{N(\mu, \sigma^2): \sigma^2 < \sigma_0^2\}$ remain invariant
- Joint suff. stat. is $(\bar{X}, \sum(X_i - \bar{X})^2)$
- A maximal invariant is $\sum(X_i - \bar{X})^2$
- The class of invariant tests consists of tests that are functions of $\sum(X_i-\bar{x})^2 \Rightarrow \frac{\sum(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$
- The pdf of $V = \sum_{i=1}^{n}(X_i - \bar{X})^2$ is given by

$$f(v) = \frac{\sigma^{-(n-1)}}{\Gamma(\frac{n-1}{2})\, 2^{\frac{(n-1)}{2}}} v^{(n-3)/2} e^{-v/2\sigma^2}, \quad v > 0$$

$\Rightarrow \{f_{\sigma^2}(v): \sigma^2 > 0\}$ has MLR in V. The UMP test is to reject H_0 if $V \leq k \Rightarrow$ The UMP invariant test is given by

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } V \leq k \\ 0 & \text{e.w.} \end{cases}$$

where $\alpha = P_{\sigma_0}(V \leq k) = P\left(\frac{V}{\sigma_0^2} \leq \frac{k}{\sigma_0^2}\right) = P\left(\chi^2_{n-1} \leq \frac{k}{\sigma_0^2}\right)$

$$\Rightarrow k = \sigma_0^2 \chi^2_{n-1}(\alpha).$$

<u>Example</u>: Let $\underset{\sim}{X}$ have pdf $f_i(x_1 - \theta, \ldots, x_n - \theta)$ under H_i $(i = 0, 1)$, $-\infty < \theta < \infty$. Let $\mathcal{G}$ be the group of translations

$g_c(\underset{\sim}{x}) = (x_1 + c, \ldots, x_n + c)$. Clearly, g induces $\bar{g}$ on Θ,

where $\bar{g}(\theta) = \theta + c$. The hypothesis testing problem remains invariant under $\mathcal{G}$.

The maximal invariant $T(\underset{\sim}{x}) = (x_1 - x_n, \cdots, x_{n-1} - x_n)$
$= (T_1, \cdots, T_{n-1})$

$\Rightarrow$ The class of invariant tests coincides with the class of tests that are functions of T.

- The pdf of T under H_i is indep. of θ and it is given by

$$f_T(t_1, \cdots, t_{n-1}) = \int_{-\infty}^{\infty} f_i(t_1 + z, \cdots, t_{n-1} + z, z)\, dz$$

- The MP test is $\phi(\underset{\sim}{t}) = \begin{cases} 1 & \text{if } \lambda(\underset{\sim}{t}) > c \\ 0 & \text{if } \lambda(\underset{\sim}{t}) < c \end{cases}$

Where

$$\lambda(\underset{\sim}{t}) = \frac{\int_{-\infty}^{\infty} f_1(t_1 + z, \cdots, t_{n-1} + z, z)\, dz}{\int_{-\infty}^{\infty} f_0(t_1 + z, \cdots, t_{n-1} + z, z)\, dz}$$ is <u>UMP invariant</u>

- A special case, $H_0: N(\theta, 1)$ vs. $H_1: C(1, \theta), \theta \in \mathbb{R}$.