Further Hypotheses Testing

The UMP tests do not exist for a wide variety of problems and restricted families are considered. A procedure leads to tests that have some desirable large-sample properties is needed [LR test].

Def.: For testing H_0 against H_1, a test of the form: reject H_0 iff $\lambda(\underset{\sim}{x}) < c$, where c is a constant and

$$\lambda(\underset{\sim}{x}) = \frac{\sup_{\theta \in H_0} f_\theta(\underset{\sim}{x})}{\sup_{\theta \in H} f_\theta(\underset{\sim}{x})} \quad \text{is called LRT}$$

where c is determined by $\alpha = \sup_{\theta \in H_0} P_\theta(\underset{\sim}{x} : \lambda(\underset{\sim}{x}) < c)$

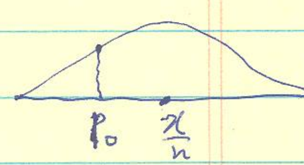
Notes

① For given α $(0 \le \alpha \le 1)$, nonrandomized Neyman-Pearson and LRT of a simple hypothesis against simple hypothesis exist and they are equivalent.

② For testing $\theta \in \Theta_0$ against $\theta \in \Theta_1$, the LRT is a function of every sufficient statistic for θ [it follows by factorization theorem].

Example: $X \sim B(n, p)$. Wish to test $H_0: p \le p_0$ vs. $H_1: p > p_0$

$$\lambda(x) = \frac{\sup_{p \le p_0} \binom{n}{x} p^x (1-p)^{n-x}}{\sup_{0 \le p \le 1} \binom{n}{x} p^x (1-p)^{n-x}}$$



$f(x) = p^x (1-p)^{n-x} \Rightarrow$ it achieves its max. at $p = \frac{x}{n}$

$$\Rightarrow \sup_{0\le p\le 1} \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x}\left(\frac{x}{n}\right)^x\left(1-\frac{x}{n}\right)^{n-x}$$

$$\sup_{p\le p_0} p^x(1-p)^{n-x} = \begin{cases} p_0^x(1-p_0)^{n-x} & \text{if } p_0 < \frac{x}{n} \\ \left(\frac{x}{n}\right)^x\left(1-\frac{x}{n}\right)^{n-x} & \text{if } p_0 \ge \frac{x}{n} \end{cases}$$

$\Rightarrow$

$$\lambda(x) = \begin{cases} \dfrac{p_0^x(1-p_0)^{n-x}}{\left(\frac{x}{n}\right)^x\left(1-\frac{x}{n}\right)^{n-x}} & \text{if } p_0 < \frac{x}{n} \quad (\text{or } x > np_0) \\ 1 & \text{if } p_0 \ge \frac{x}{n} \quad (\text{or } x \le np_0) \end{cases}$$

$$= \begin{cases} \left(\dfrac{p_0}{(x/n)}\right)^x \left(\dfrac{1-x/n}{1-p_0}\right)^x \cdot \text{const.} & \text{if } p_0 < \frac{x}{n} \\ 1 & \text{if } p_0 \ge \frac{x}{n} \end{cases}$$

$\Rightarrow$ $\lambda(x) \le 1$ for $x > np_0$ and
$\lambda(x) = 1$ for $x \le np_0$

$\Rightarrow$ $\lambda(x) < c \Leftrightarrow X > c'$ with $\alpha = \sup_{p \le p_0} P(X > c')$

<u>Example</u>: Consider the problem of testing $H_0: \mu = \mu_0$ against $H_1: \mu \ne \mu_0$ in a sampling from $N(\mu, \sigma^2)$ where μ & σ^2 are unknown.
In this case,

$$\Theta_0 = \{(\mu_0, \sigma^2) : \sigma^2 > 0\}$$
$$\Theta_1 = \{(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0\}$$

$$\sup_{\theta\in\Theta_0} f_\theta(\underline{x}) = \sup_{\sigma^2>0}\left[\frac{1}{(\sqrt{2\pi}\,\sigma)^n} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu_0)^2}\right]$$

- The MLE of σ^2 when sampling from $N(\mu_0, \sigma^2)$

$$\log L \propto -n \log \sigma - \frac{1}{2\sigma^2} \sum (X_i - \mu_0)^2$$

$$\frac{-n}{\sigma} + \frac{1}{\sigma^3} \sum (X_i - \mu_0)^2 = 0 \Rightarrow \hat{\sigma}_0^2 = \frac{\sum (X_i - \mu_0)^2}{n}$$

$$\Rightarrow \sup_{\Theta_0} f_\theta(\underset{\sim}{x}) = L(\mu_0, \hat{\sigma}_0^2) = \frac{1}{(\sqrt{2\pi})^n \hat{\sigma}_0^n} e^{-\frac{n}{2}}.$$

- Under Θ (unrestrected estimation): The MLEs of μ & σ^2 when both are unknown,

$$\hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = \frac{\sum (X_i - \bar{X})^2}{n}$$

$$\Rightarrow$$

$$\sup_{\theta \in \Theta} f_\theta(\underset{\sim}{x}) = \sup_{\mu, \sigma^2} \left[\frac{1}{(\sqrt{2\pi}\,\sigma)^n} e^{-\frac{1}{2\sigma^2} \sum (X_i - \mu)^2} \right]$$

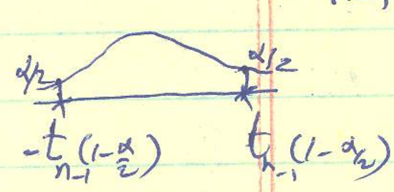
$$= \frac{1}{(2\pi/n)^{n/2}} \frac{e^{-\frac{n}{2}}}{\left[\sum (X_i - \bar{X})^2 \right]^{n/2}}$$

$$\Rightarrow \lambda(\underset{\sim}{x}) = \left[\frac{\sum (X_i - \bar{X})^2}{\sum (X_i - \mu_0)^2} \right]^{n/2} = \left[\frac{1}{1 + \frac{n(\bar{X} - \mu_0)^2}{\sum (X_i - \bar{X})^2}} \right]^{n/2}$$

$$\Rightarrow \text{LRT is to rej. } H_0 \text{ if } \lambda(\underset{\sim}{x}) < c \Leftrightarrow \frac{n(\bar{X} - \mu_0)^2}{\sum (X_i - \bar{X})^2} > c'$$

$$\Leftrightarrow \left| \frac{\bar{X} - \mu_0}{\sum (X_i - \bar{X})^2} \right| > c' \Leftrightarrow \left| \frac{\sqrt{n}(\bar{X} - \mu_0)}{S} \right| > c^*, \quad S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1}$$

where $\frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \sqrt{n} \frac{(\bar{X} - \mu_0)}{S} \sim t_{n-1}$



$$c^* = t_{n-1}(1 - \tfrac{\alpha}{2})$$

<u>Note</u>: This test is also UMP unbiased test.

Theorem: Under some regularity conditions on $f_\theta(\underset{\sim}{x})$, $-2\log\lambda(\underset{\sim}{x})$ is asymptotically χ^2_r, $r = \#$ of indep. parameters in Θ $- \#$ of indep. parameters in Θ_0

Previous Example: $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$

$$\lambda(\underset{\sim}{x}) = \left\{1 + \frac{n(\bar{x}-\mu_0)^2}{\sum(x_i-\bar{x})^2}\right\}^{-n/2}$$

Under H_0: # of parameters unspecified (namely σ^2) $= 1$

Under Θ: " " " " (μ, σ^2) $= 2$

$$\Rightarrow -2\log\lambda(\underset{\sim}{x}) = n\log\left\{1 + \frac{n(\bar{x}-\mu_0)^2}{\sum(x_i-\bar{x})^2}\right\}$$

$$\xrightarrow{\mathcal{D}} \chi^2_{2-1} = \chi^2_1 .$$

Confidence Estimation

For given α $(0<\alpha<1)$ if $P(\underset{\sim}{\theta} \in S(\underset{\sim}{x})) \geq 1-\alpha$, $\forall\, \underset{\sim}{\theta} \in \Omega$, then $S(\underset{\sim}{x})$ is $(1-\alpha)$ confidence region. Note $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_k)$

For $k=1$, the lower conf. bdd (LCB) is $S(\underset{\sim}{x}) = \{\theta : \theta \geq \underline{\theta}(\underset{\sim}{x})\}$

" upper " " (UCB) is $S(\underset{\sim}{x}) = \{\theta : \theta \leq \bar{\theta}(\underset{\sim}{x})\}$

If $S(\underset{\sim}{x})$ is of the form $S(\underset{\sim}{x}) = (\underline{\theta}(\underset{\sim}{x}), \bar{\theta}(\underset{\sim}{x}))$, we will call it confidence interval (CI) at confidence level $1-\alpha$, provided

$$P(\underline{\theta}(\underset{\sim}{x}) < \theta < \bar{\theta}(\underset{\sim}{x})) \geq 1-\alpha, \quad \forall\, \theta.$$

__Example__: Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, where σ^2 is known. The Pivot Statistic is $V = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$. Its dist. is free of μ.

$$1-\alpha = P\left(-z_{1-\frac{\alpha}{2}} < \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < z_{1-\frac{\alpha}{2}}\right)$$

$$= P\left(\bar{X} - z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right)$$

$\Rightarrow (1-\alpha)100\%$ C.I. for μ is $\left(\bar{X} - z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}},\ \bar{X} + z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right)$

If σ is unknown, $V_1 = \frac{\sqrt{n}(\bar{X}-\mu)}{S}$, $V_2 = \frac{(n-1)S^2}{\sigma^2}$ are pivots

$$V_1 = \frac{(\bar{X}-\mu)/\sigma/\sqrt{n}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}} \equiv \frac{N(0,1)}{\sqrt{\chi^2_{n-1}/(n-1)}} \sim t_{n-1}$$

$$V_2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

C.I. for μ: $\left(\bar{X} - t_{n-1}(1-\alpha/2)\frac{S}{\sqrt{n}},\ \bar{X} + t_{n-1}(1-\frac{\alpha}{2})\frac{S}{\sqrt{n}}\right)$

C.I. for σ^2: $1-\alpha = P\left(\chi^2_{n-1}(\alpha/2) < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1}(1-\alpha/2)\right)$

$$\Rightarrow \left(\frac{(n-1)S^2}{\chi^2_{n-1}(1-\frac{\alpha}{2})},\ \frac{(n-1)S^2}{\chi^2_{n-1}(\alpha/2)}\right)$$

Theorem: If $X_1, \dots, X_n \overset{r.s.}{\sim}$ from continuous dist. $F(x;\theta)$. If $F(x;\theta)$ is monotone in θ, then the statistic $Q = -2\sum^n \log F(X_i;\theta)$ is a pivotal.

$$F(X;\theta) \sim U(0,1) \Rightarrow -2\log F(X;\theta) \sim \chi^2_2$$
$$\Rightarrow Q \sim \chi^2_{2n}$$

Example: Let $X_1, \dots, X_n \overset{r.s.}{\sim} U(0,\theta)$. Find $(1-\alpha)100\%$ C.I. for θ

- $T = X_{(n)}$ MLE of θ
- $\frac{X_{(n)}}{\theta} \sim \text{Beta}(n,1)$ since $P(X_{(n)} \le y) = P(X_1 \le y, \dots, X_n \le y) = [F(y)]^n$
  $= \left(\frac{y}{\theta}\right)^n,\ 0<y<\theta$

  $G(t) = \left(\frac{t}{\theta}\right)^n \Rightarrow G(T) = \left(\frac{T}{\theta}\right)^n \sim U(0,1)$

- Consider $1-\alpha = P(A \le Q \le B)$

  $\alpha/2 \quad 1-\alpha \quad \alpha/2$

  $A = \alpha/2 \quad B = 1-\alpha/2$

$$= P\left(\frac{\alpha}{2} \le \left(\frac{T}{\theta}\right)^n \le 1-\frac{\alpha}{2}\right)$$
$$= P\left(\left(\frac{\alpha}{2}\right)^{1/n} \le \frac{T}{\theta} \le (1-\alpha/2)^{1/n}\right)$$
$$= P\left(\frac{T}{(1-\frac{\alpha}{2})^{1/n}} < \theta < \frac{T}{(\alpha/2)^{1/n}}\right)$$

$\Rightarrow \left(\frac{T}{(1-\frac{\alpha}{2})^{1/n}},\ \frac{T}{(\alpha/2)^{1/n}}\right)$ is $(1-\alpha)100\%$ C.I. for θ.

Example: Let $X_1, \ldots, X_n \overset{r.s.}{\sim} Exp(\theta)$ - Find $(1-\alpha)100\%$ C.I. for θ

$$F(x;\theta) = 1 - e^{-x/\theta}$$

$$1 - F(x;\theta) = e^{-x/\theta}$$

$$\Rightarrow Q = -2\sum \log(1 - F(X_i;\theta)) = \frac{2}{\theta}\sum^n X_i \sim \chi^2_{2n}$$

$$1-\alpha = P\left(\chi^2_{2n}(\alpha/2) \le \frac{2\sum X_i}{\theta} \le \chi^2_{2n}(1-\alpha/2)\right)$$

$$= P\left(\frac{2\sum X_i}{\chi^2_{2n}(1-\alpha/2)} < \theta < \frac{2\sum X_i}{\chi^2_{2n}(\alpha/2)}\right)$$

Example: Let $X_1, X_2, \ldots, X_n \overset{r.s.}{\sim} f(x;\theta) = \theta x^{\theta-1}, 0<x<1, \theta>0$. Obtain $(1-\alpha)100\%$ C.I. for θ.

$$L = \theta^n \left(\Pi X_i\right)^{\theta-1} \Rightarrow \log L \propto -n\log\theta + (\theta-1)\sum \log X_i$$

$$\Rightarrow \quad \hat{\theta} = \frac{n}{\sum -\log X_i}$$

$$- \quad Y = \theta \sum -\log X_i \overset{d}{=} G(n,1) \Rightarrow 2\theta \sum^n -\log X_i \sim \chi^2_{2n} \text{ "Pivot"}$$

$$\Rightarrow \left(\frac{\chi^2_{2n}(\alpha/2)}{2\sum -\log X_i}, \frac{\chi^2_{2n}(1-\alpha/2)}{2\sum -\log X_i}\right) \text{ is } (1-\alpha)100\% \text{ C.I. for } \theta.$$

? Can we obtain asymptotic C.I.?

Large Sample Theory

Def.: $Y_n \xrightarrow{P} c$ iff $P(|Y_n - c| > \varepsilon) \to 0$ as $n \to \infty$. A sequence of estimators δ_n of $g(\theta)$ is consistent if for every $\theta \in \Theta$, $\delta_n \xrightarrow{P} g(\theta)$.

Lemma: A sufficient condition for Y_n to converge in probability to c is that $E(Y_n - c)^2 \to 0$

Proof: $P(|Y_n - c| > \varepsilon) \le \dfrac{E(Y_n - c)^2}{\varepsilon^2} \to 0$ "Chebyshev's Inequality"

Theorem:

(i) Let δ_n be a seq. of estimators of $g(\theta)$ with risk, $R(\theta, \delta_n)$ where $R(\theta, \delta_n) = E(\delta_n - \theta)^2$. Then $R(\theta, \delta_n) \to 0, \forall \theta$ $\Rightarrow \delta_n$ is consistent for $g(\theta)$.

(ii) Equivalent to the above result, δ_n is consistent if $b_n(\theta) \to 0$ and $Var(\delta_n) \to 0, \forall \theta$ where b_n is the bias of δ_n.

(iii) In particular, δ_n is consistent if it is unbiased for each n and if $Var(\delta_n) \to 0, \forall \theta$.

Example: $X_1, \ldots, X_n \overset{iid}{\sim}$ some dist. with finite mean μ and Var. σ^2

$$S_n^2 = \frac{\sum (X_i - \bar{X})^2}{n-1} = \frac{n}{n-1} \frac{\sum (X_i - \bar{X})^2}{n} \xrightarrow[WLLN]{P} 1 \cdot E(X_i - \bar{X})^2$$

$$= Var(X_i) + Var(\bar{X}) - 2\,Cov(X_i, \bar{X})$$

$$= \sigma^2 + \frac{\sigma^2}{n} - \frac{2}{n}\sigma^2 = \sigma^2 - \frac{\sigma^2}{n} \to \sigma^2$$

$\Rightarrow S_n^2$ is a consistent estimator of σ^2.

Central Limit Theorem (CLT)

Let X_i $(i=1,\ldots,n)$ be iid with $EX_i = \xi$ & $Var(X_i) = \sigma^2 < \infty$.
Then $\sqrt{n}(\bar{X}-\xi) \xrightarrow{\mathcal{D}} N(0,\sigma^2)$ or $\sqrt{n}\frac{(\bar{X}-\xi)}{\sigma} \xrightarrow{\mathcal{D}} N(0,1)$.

Theorem: If $Y_n \xrightarrow{\mathcal{D}} Y$ and A_n & B_n tend in probability to a & b, respectively, then $A_n + B_n Y_n \xrightarrow{\mathcal{D}} a + bY$.

Theorem: If $\sqrt{n}(T_n - \theta) \xrightarrow{\mathcal{D}} N(0,\tau^2)$, then

$$\sqrt{n}(f(T_n)-f(\theta)) \xrightarrow{\mathcal{D}} N(0, \tau^2[f'(\theta)]^2)$$

provided $f'(\theta)$ exists & is not zero (Delta method).

Example: Let $X_1,\ldots,X_n \overset{iid}{\sim} N(\theta,\sigma^2)$

$\sqrt{n}(\bar{X}-\theta) \longrightarrow N(0,\sigma^2)$ by CLT

$\sqrt{n}(\bar{X}^2-\theta^2) \longrightarrow N(0,\sigma^2\cdot(4\theta^2))$ Delta method

Multivariate CLT: Let $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots, \underset{\sim}{X}_n$ be iid with mean $\underset{\sim}{\xi} = (\xi_1,\ldots,\xi_r)$ and Covariance matrix $\Sigma = (\sigma_{ij})$, where $\underset{\sim}{X}_i = (X_{i1},\ldots,X_{ir})$. Then

$$[\sqrt{n}(\bar{X}_1-\xi_1), \ldots, \sqrt{n}(\bar{X}_r-\xi_r)] \xrightarrow{\mathcal{D}} MN(\underset{\sim}{0}, \Sigma)$$

Asymptotic optimality

- $\sqrt{n}(\delta_n - g(\theta)) \xrightarrow{\mathcal{D}} N(0,v)$

  By Cramer-Rao (CR) bound, $Var(\delta_n) \geq \frac{[g'(\theta)]^2}{n I_1(\theta)}$ for any unbiased estimate of $g(\theta)$.
- If $\exists\, \delta_n \ni Var(\delta_n) = [g'(\theta)]^2/nI_1(\theta)$, then δ_n is asymptotically efficient
- If $g(\theta)=\theta$, $v(\theta) \geq \frac{1}{nI_1}$, where I_1 is the information in X_1 about θ.

  $I_1(\theta) = E\left[\frac{\partial \log P_\theta(x)}{\partial\theta}\right]^2$.

Information Inequalities

$\Theta \subset \mathbb{R}$, $\underset{\sim}{X} \sim P_\theta$: common support

$$\frac{\dot{P}_\theta}{P_\theta} = \frac{\partial}{\partial\theta} \log P_\theta(x)$$

- $E\left[\frac{\partial}{\partial\theta} \log P_\theta(x)\right] = 0$, since $LHS = \int_x \frac{\dot{P}_\theta}{P_\theta} P_\theta \, dx = \int \dot{P}_\theta(x)\, dx$
$$= \frac{\partial}{\partial\theta} \int P_\theta(x)\, dx = 0$$

- If $\frac{\partial^2}{\partial\theta^2} \log P_\theta(x)$ exists, $\forall x \in$ support, $\theta \in \Theta$
$$E\left(\frac{\partial}{\partial\theta} \log P_\theta(x)\right)^2 = -E\left(\frac{\partial^2 \log P_\theta(x)}{\partial\theta^2}\right) = I_1(\theta)$$
$I(\theta)$ = Fisher inf. based on random sample of size n
$= n I_1(\theta)$, where $I_1(\theta)$ is FI based on X_1.

- To show $Var(\delta(x)) \geq \frac{[E\dot{\delta}(x)]^2}{n I_1} = \frac{(g'(\theta))^2}{n I_1}$

$$Corr\left(\delta(x), \frac{\partial \log P_\theta(x)}{\partial\theta}\right) \leq 1$$

$$(LHS)^2 = \frac{Cov\left(\delta(x), \frac{\partial}{\partial\theta} \log P_\theta(x)\right)}{\sqrt{Var\,\delta}\ \sqrt{Var\left(\frac{\partial \log P_\theta(x)}{\partial\theta}\right)}} = \frac{E\left[\delta(x) \cdot \frac{\partial \log P_\theta(x)}{\partial\theta}\right]}{\sqrt{Var\,\delta}\ \sqrt{Var\, \frac{\partial \log P_\theta(x)}{\partial\theta}}}$$

$$E\left[\delta(x) \cdot \frac{\partial \log P_\theta(x)}{\partial\theta}\right] = \int \delta(x) \cdot \frac{\dot{P}_\theta(x)}{P_\theta(x)} \cdot P_\theta(x)\, dx = \frac{\partial}{\partial\theta} E\delta(x)$$
$$= g(\theta)$$

$$Var\left(\frac{\partial \log P_\theta(x)}{\partial\theta}\right) = E\left(\frac{\partial \log P_\theta(x)}{\partial\theta}\right)^2 = n I_1$$

$$\Rightarrow LHS = \frac{(g'(\theta))^2}{n\, Var(\delta) \cdot I_1(\theta)} \leq 1 \Rightarrow Var(\delta(\underset{\sim}{x})) \geq \frac{(g'(\theta))^2}{n I_1}.$$

Asymptotic C.I.

Let $\hat{\theta}_n$ be MLE of θ based on n obs.'s. As $n \to \infty$, $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, I_1^{-1}(\theta)) \Rightarrow \sqrt{nI_1}\,(\hat{\theta}_n - \theta) \longrightarrow N(0,1)$

$\sqrt{nI_1}\,(\hat{\theta}_n - \theta)$ is called an approx. pivot.

$$\Rightarrow P\left(-z_{1-\alpha/2} \le \sqrt{nI_1}\,(\hat{\theta}_n - \theta) \le z_{1-\alpha/2}\right) = 1-\alpha$$

$$\Rightarrow P\left(\hat{\theta}_n - z_{1-\frac{\alpha}{2}} \frac{1}{\sqrt{nI_1}} \le \theta \le \hat{\theta}_n + z_{1-\frac{\alpha}{2}} \frac{1}{\sqrt{nI_1}}\right) = 1-\alpha$$

Example: Let X be either 1 or 2 according to the toss of a fair coin, and $Y \mid X = x \sim N(\theta, x)$.

$$\Rightarrow f_\theta(x,y) = \frac{1}{2\sqrt{2\pi x}} e^{-\frac{1}{2x}(y-\theta)^2}$$

$$\log f_\theta(x,y) \propto -\frac{1}{2x}(y-\theta)^2 \Rightarrow I_1(\theta) = -E\frac{\partial^2 \log f}{\partial \theta^2} = E\left(\frac{1}{X}\right) = 3/4$$

If $(X_1, Y_1), \dots, (X_n, Y_n)$ is a random sample from this dist., then

$$\log L = \sum_{i=1}^{n} \log f(X_i, Y_i) = \sum_{i=1}^{n}\left[\frac{-1}{2X_i}(Y_i - \theta)^2 - \frac{1}{2}\log(8\pi X_i)\right]$$

$$(\log L)' = \sum_{i=1}^{n} \frac{Y_i - \theta}{X_i} = 0$$

$$(\log L)'' = -\sum^{n} \frac{1}{X_i}$$

$\Rightarrow (1-\alpha)100\%$ C.I. for θ (asymptotic) is $\left(\hat{\theta}_n - z_{1-\alpha/2}\sqrt{\frac{4}{3n}},\ \hat{\theta}_n + z_{1-\alpha/2}\sqrt{\frac{4}{3n}}\right)$

OR

$$n\widehat{I_1(\theta)} = \sum^{n} \frac{1}{X_i}$$

$\Rightarrow (1-\alpha)100\%$ approx. C.I. for θ is

$$\left(\hat{\theta}_n - z_{1-\frac{\alpha}{2}} \frac{1}{\sum \frac{1}{X_i}},\ \hat{\theta}_n + z_{1-\alpha/2} \frac{1}{\sum \frac{1}{X_i}}\right)$$